J. Fluid Mech. (1991), vol. 224, pp. 551–578
Printed in Great Britain

Nonlinear evolution of subsonic and supersonic disturbances on a compressible free shear layer

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(Received 18 May 1990 and in revised form 24 August 1990)

We consider the effects of a nonlinear–non-equilibrium–viscous critical layer on the spatial evolution of subsonic and supersonic instability modes on a compressible free shear layer. It is shown that the instability wave amplitude is governed by an integro-differential equation with cubic-type nonlinearity. Numerical and asymptotic solutions to this equation show that the amplitude either ends in a singularity at a finite downstream distance or reaches an equilibrium value, depending on the Prandtl number, viscosity law, viscous parameter and a real parameter which is determined by the linear inviscid stability theory. A necessary condition for the existence of the equilibrium solution is derived, and whether or not this condition is met is determined numerically for a wide range of physical parameters including both subsonic and supersonic disturbances. It is found that no equilibrium solution exists for the subsonic modes unless the temperature ratio of the low-to high-speed streams exceeds a critical value, while equilibrium solutions for the most rapidly growing supersonic mode exist over most of the parameter range examined.

1. Introduction

The stability of compressible flows has received renewed attention lately, primarily owing to interest in controlling mixing in high-speed propulsion systems.

The linear, inviscid, spatial stability properties of a compressible shear layer have been studied by Gropengeisser (1969) and more recently by Jackson & Grosch (1989*a*, *b*). The evolution of the mode with the largest (linear) growth rate is normally of greatest interest since this mode will quickly come to dominate the flow and be responsible for the onset of nonlinearity. The linear calculations mentioned above confirmed earlier work of Dunn & Lin (1955) which indicated that oblique waves are important for compressible flows. In particular, the most rapidly growing mode is oblique when the Mach number exceeds about 2.

The absolute/convective nature of the instability of a compressible shear layer has been examined by Pavithran & Redekopp (1989) and Jackson & Grosch (1990). They found that shear layers with negative free-stream velocity ratios can be absolutely unstable at subsonic Mach numbers. Pavithran & Redekopp (1989) also found that, with severe cooling of the low-speed stream, even a coflowing shear layer will become absolutely unstable. Although neither of these works considered supersonic flows there is a clear trend towards convective instability as the Mach number increases.

The early stage of the evolution of an instability wave generated by a small-amplitude, single-frequency excitation of a convectively unstable plane shear layer is well described by linear, inviscid spatial theory. Spreading of the mean flow on the long (compared with the wavelength) viscous lengthscale serves to reduce the local growth rate while the instability wave amplitude continues to grow. When the amplitude becomes sufficiently large and the growth rate sufficiently small, nonlinear effects become important in the 'critical layer' surrounding the point where the phase speed is equal to the mean flow velocity. When this occurs nonlinear effects within the critical layer determine the growth rate of the instability wave in the outer region which is otherwise governed by linear dynamics.

Goldstein & Leib (1989, which we will refer to as I) showed that for a compressible shear layer nonlinear critical-layer effects occur at smaller amplitudes than for the two-dimensional incompressible case considered by Goldstein & Leib (1988) and Goldstein & Hultgren (1988) because the temperature fluctuations have an algebraic singularity at the critical level. The critical-layer dynamics are weakly nonlinear in the compressible case in that the nonlinearity appears only as inhomogeneous terms in the critical-layer vorticity and energy equations. In I an analysis was carried out for oblique subsonic modes whose critical layer coincides with a generalized inflexion point and hence are regular there. It was shown that the instability wave amplitude is governed by an integro-differential equation of the form found by Hickernell (1984) with coefficients which must be calculated numerically from the linear solution. Numerical solutions to the amplitude equation for the inviscid and viscous cases were found to behave quite differently, with the former ending in a singularity at a finite downstream distance and an equilibrium solution possible for the latter when the coefficients lie in certain regions of the parameter space. A calculation of the coefficients for one particular case showed that the equilibrium solution is a physically realizable one.

In this paper we generalize the analysis of I to the case of supersonic modes with singular critical layers. While the analysis of Hickernell (1984) considered modes with singular critical layers, these modes are not the most rapidly growing linear instabilities and therefore would probably not be observed in practice. For the compressible shear layer considered here, however, the numerical calculations of Gropengeisser (1969) and Jackson & Grosch (1989*a*, *b*) have demonstrated that at sufficiently high Mach numbers the most rapidly growing mode is a supersonic mode with a singular critical layer. There are two such modes which are usually characterized according to their phase speeds. The 'fast' mode has a phase speed which goes to unity as the Mach number approaches infinity while that of the 'slow' mode goes to zero.

We further remove the restrictions of unit Prandtl number and linear viscosity – temperature relation of I and assess their effects on the critical-layer solution. Also we derive an integral condition for the coefficients appearing in the amplitude equation and provide numerical results for them over a wide range of physical parameters, which includes the subsonic modes considered in I as well as the supersonic modes, to determine the types of solutions possible for cases of physical interest. Since these results depend to some extent on the mean flow model chosen, we compare results obtained for a number of models to determine the effect on our conclusions.

In §2 the problem is formulated and the outer solution is obtained. As in I we use the pressure as the basic dependent variable, with the 'Squire coordinates' in the directions along and normal to the direction of wave propagation as independent variables. In these coordinates the solution is independent of the 'spanwise' coordinate. The primary result of the outer solution is the 'streamwise' velocity jump across the critical layer.

In §3 the solution in the critical layer is obtained in terms of the 'spanwise vorticity' and the temperature. Matching the velocity jumps calculated from the inner and outer solutions yields the Hickernell (1984)-type evolution equation for the amplitude evolution.

In §4 asymptotic solutions to the amplitude equation are derived and a necessary condition for the existence of an equilibrium solution is obtained. In §5 we present and discuss the numerical results.

2. Formulation and outer solution

We consider the flow of an ideal gas between parallel streams with uniform temperatures $T^{(1)}$, $T^{(2)}$ and velocities $U^{(1)} > U^{(2)}$. The flow quantities in the high-speed stream (denoted by superscript 1) are used as reference quantities.

We define the Mach number and Reynolds number as

$$M = U^{(1)}/C^{(1)} \tag{2.1}$$

and

$$Re = U^{(1)}\delta/\nu^{(1)}, \tag{2.2}$$

respectively, where δ is a measure of the shear-layer thickness,

$$C^{(1)} = (k\mathscr{R}T^{(1)})^{\frac{1}{2}} \tag{2.3}$$

and $\nu^{(1)}$ are the speed of sound and kinematic viscosity in the high-speed stream respectively, k is the isentropic exponent of the gas and $\mathscr{R}$ is the gas constant.

The local Prandtl number, which we will assume to be an order-one constant, is defined as

$$\sigma_0 = \mu c_p/\kappa, \tag{2.4}$$

where μ, c_p and κ are the normalized viscosity, specific heat and thermal conductivity, respectively. We suppose that the normalized viscosity has a power-law dependence on the temperature, namely

$$\mu = T^n, \tag{2.5}$$

where n is a constant.

As in I we consider an oblique (three-dimensional) instability wave growing in its direction of propagation on the otherwise steady shear flow formed by the two streams. The linear neutral Strouhal number, streamwise wavenumber and spanwise wavenumber of the instability wave are denoted by S_0, α_0 and β_0 respectively. As shown in I nonlinear effects first become important in the critical layer when the local Strouhal number S differs from the linear neutral value by an amount of order $\epsilon^{\frac{2}{5}}$ so that

$$S = S_0 + \epsilon^{\frac{2}{5}} S_1, \tag{2.6}$$

where $S_1 < 0$ is an order-one constant and $\epsilon \ll 1$ is the order of the instability wave amplitude. We fix the origin of the x (streamwise), y (cross-stream), z (spanwise) coordinate system to be in this nonlinear region. Time is denoted by t.

We wish to consider the case where viscous effects are of the same order as the nonlinearity within the critical layer. For this to occur the Reynolds number must scale with the amplitude as

$$Re \sim \epsilon^{-\frac{6}{5}}. \tag{2.7}$$

If Re is chosen to be smaller than $O(\epsilon^{-\frac{6}{5}})$ the critical layer will be dominated by viscous effects; if it is larger viscosity will not influence the solution. Then the 'viscous parameter'

$$\lambda = \frac{1}{Re\,\epsilon^{\frac{6}{5}}} \tag{2.8}$$

is an order-one constant which gives a measure of the importance of the viscous effects. The inviscid case is obtained by setting $\lambda \equiv 0$.

It is convenient to work in the oblique coordinate system

$$\bar{\xi} = x\cos\theta + z\sin\theta - U_c\cos\theta t, \tag{2.9}$$

$$\bar{z} = -x\sin\theta + z\cos\theta, \tag{2.10}$$

where $\theta = \tan^{-1}\beta_0/\alpha_0$ is the angle the wave makes with the mean flow direction, $U_c\cos\theta = S_0/\bar{\alpha}$ is the neutral phase speed, and

$$\bar{\alpha} = (\alpha_0^2 + \beta_0^2)^{\frac{1}{2}}. \tag{2.11}$$

The velocity components $\{\bar{u}, v, \bar{w}\}$ in these coordinates are related to those in the original coordinates by

$$\bar{u} = u\cos\theta + w\sin\theta - U_c\cos\theta, \tag{2.12}$$

$$\bar{w} = -u\sin\theta + w\cos\theta. \tag{2.13}$$

2.1. *The slowly varying mean flow*

We assume that the mean pressure is constant across the shear layer so that for an ideal gas the mean density $\bar{\rho}$ and temperature $\bar{T}$ are related by

$$\bar{\rho}\bar{T} = 1. \tag{2.14}$$

In the local nonlinear region the instability wave evolves in its propagation direction on the long lengthscale

$$\bar{x}_1 = \epsilon^{\frac{2}{5}}(x\cos\theta + z\sin\theta), \tag{2.15}$$

while the mean flow spreads on the longer viscous lengthscale

$$x_2 = x/Re = \epsilon^{\frac{4}{5}}\lambda\bar{x}_1\cos\theta + \epsilon^{\frac{6}{5}}\bar{z}\sin\theta. \tag{2.16}$$

Locally then, we may expand the mean flow in a Taylor series about the origin of the streamwise coordinate as

$$\bar{U}(y, x_2) = U(y)\cos\theta + a(y)\cos\theta\{\epsilon^{\frac{4}{5}}\lambda\bar{x}_1\cos\theta + \epsilon^{\frac{6}{5}}\lambda\bar{z}\sin\theta\} + \dots, \tag{2.17}$$

$$\hat{V}(y, x_2) = \hat{V}_0(y) + b(y)\{\epsilon^{\frac{4}{5}}\lambda\bar{x}_1\cos\theta + \epsilon^{\frac{6}{5}}\lambda\bar{z}\sin\theta\} + \dots, \tag{2.18}$$

$$\bar{W}(y, x_2) = -(U(y) + U_c)\sin\theta - a(y)\sin\theta\{\epsilon^{\frac{4}{5}}\lambda\bar{x}_1\cos\theta + \epsilon^{\frac{6}{5}}\lambda\bar{z}\sin\theta\} + \dots, \tag{2.19}$$

$$\bar{T}(y, x_2) = T_0(y) + d(y)\{\epsilon^{\frac{4}{5}}\lambda\bar{x}_1\cos\theta + \epsilon^{\frac{6}{5}}\lambda\bar{z}\sin\theta\} + \dots, \tag{2.20}$$

where $U(y) + U_c$ is the mean velocity (in the stationary frame) at the origin, $\hat{V} = Re\,\bar{V}$ and the functions a, b, d can be determined from the boundary-layer equations. Only the leading terms in the expansions (2.17)–(2.20) influence the velocity jump across the critical layer so that for our purposes it is enough to specify a velocity and temperature profile at the origin.

2.2. *The solution outside the critical layer*

The solution outside the critical layer expands like

$$\bar{u} = U(y)\cos\theta + \epsilon^{\frac{4}{5}}a(y)\,\lambda\bar{x}_1\cos^2\theta + \epsilon\,\mathrm{Re}\,[F(y)\,A^{\dagger}(\bar{x}_1)\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{6}{5}}\bar{u}_2 + \dots, \tag{2.21}$$

$$v = -\epsilon\bar{\alpha}\,\mathrm{Re}\,[\mathrm{i}\Phi(y)\,A^{\dagger}(\bar{x}_1)\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{6}{5}}v_2 + \dots, \tag{2.22}$$

$$\bar{w} = -\sin\theta(U_c + U(y)) - \epsilon^{\frac{4}{5}}a(y)\,\lambda\bar{x}_1\sin\theta\cos\theta + \epsilon\,\mathrm{Re}\,[\Psi(y)\,A^{\dagger}(\bar{x}_1)\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{6}{5}}\bar{w}_2 + \dots, \tag{2.23}$$

$$T = T_0(y) + \epsilon^{\frac{4}{5}}d(y)\,\lambda\bar{x}_1\cos\theta + \epsilon\,\mathrm{Re}\,[\Theta(y)\,A^{\dagger}(\bar{x}_1)\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{6}{5}}\theta_2 + \dots, \tag{2.24}$$

$$p = 1 + \epsilon kM^2\cos\theta\,\mathrm{Re}\,[\Pi(y)\,A^{\dagger}(\bar{x}_1)\,\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{6}{5}}p_2 + \dots, \tag{2.25}$$

where
$$\zeta = \bar{\xi} - S_1 \epsilon^{\frac{2}{5}} t/\bar{\alpha} \tag{2.26}$$
and the unknown function $A^\dagger$ will be determined by matching with the flow in the critical layer. Matching with the upstream linear solution requires
$$A^\dagger \sim a^\dagger \exp(-S_1 U_c' \bar{\kappa} \bar{x}_1/2) \quad \text{as} \quad \bar{x}_1 \to -\infty, \tag{2.27}$$
where $\mathrm{i}\frac{1}{2} S_t U_c' \bar{\kappa}$ is a scaled wavenumber whose imaginary part is minus the linear growth rate of the instability wave.

The $O(\epsilon^{\frac{6}{5}})$ terms were introduced in order to match with the critical-layer solution and accounts for the slowly varying mean flow terms.

To the required order of approximation the outer solution is linear and the functions F, Φ, Ψ, Θ and Π are determined by
$$\tilde{\alpha}^2 (U-c)^2 F = -T_0[\tilde{\alpha}^2 (U-c)\Pi + U' \mathrm{D}\Pi], \tag{2.28}$$
$$\tilde{\alpha}^2 (U-c)\Phi = -T_0 \mathrm{D}\Pi, \tag{2.29}$$
$$\tilde{\alpha}^2 (U-c)^2 \Psi = T_0 U' \tan\theta \mathrm{D}\Pi, \tag{2.30}$$
$$\cos\theta (U-c)\Theta = T_0' \Phi + (k-1) M^2 \cos^2\theta (U-c) T_0 \Pi, \tag{2.31}$$
$$(U-c)^2 \mathrm{D} \frac{T_0}{(U-c)^2} \mathrm{D}\Pi - \tilde{\alpha}^2 [T_0 - M^2 \cos^2\theta (U-c)^2] \Pi = 0, \tag{2.32}$$
where
$$\tilde{\alpha} = \bar{\alpha} + \frac{\epsilon^{\frac{2}{5}}}{\mathrm{i}A^\dagger} \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1}, \tag{2.33}$$
$$c = \epsilon^{\frac{2}{5}} \left(\frac{S_1}{\alpha} - U_c \frac{\mathrm{d}A^\dagger/\mathrm{d}\bar{x}_1}{\mathrm{i}\bar{\alpha} A^\dagger} \right), \tag{2.34}$$
the primes denote differentiation with respect to the relevant variables and we have put
$$\mathrm{D} = \frac{\partial}{\partial y}. \tag{2.35}$$

These functions have expansions of the form $F = F_1 + \epsilon^{\frac{2}{5}} F_3 + \dots$, etc. Substituting these into (2.28)–(2.34) we obtain
$$\mathrm{L}\Pi_1 = 0, \tag{2.36}$$
$$\begin{aligned} \mathrm{i}\bar{\alpha} U A^\dagger \mathrm{L}\Pi_3 = {} & 2U\bar{\alpha}^2 \left(1 - \frac{M^2 U^2}{T_0}\cos^2\theta\right) \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} \Pi_1 \\ & - \sec\theta \left(U_c \cos\theta \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^\dagger \right) \left[\frac{1}{T_0} \frac{\mathrm{d}}{\mathrm{d}y} T_0 \frac{\mathrm{d}}{\mathrm{d}y} - \bar{\alpha}^2 \left(1 - 3\frac{U^2 M^2}{T_0}\cos^2\theta\right)\right] \Pi_1, \end{aligned} \tag{2.37}$$
$$\bar{\alpha}^2 F_1 = -\frac{T_0}{U} \left(\bar{\alpha}^2 \Pi_1 + \frac{U'}{U} \Pi_1' \right), \tag{2.38}$$
$$\begin{aligned} \bar{\alpha}^2 A^\dagger F_3 = {} & -\frac{T_0}{U} \Bigg[A^\dagger \left(\bar{\alpha}^2 \Pi_3 + \frac{U'}{U} \Pi_3' \right) \\ & - \frac{\mathrm{i}\bar{\alpha}}{\cos\theta} \left(U_c \cos\theta \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^\dagger \right) \left(\frac{\Pi_1}{U} + \frac{2F_1}{T_0} \right) - 2\mathrm{i}\bar{\alpha} \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} \left(\Pi_1 + \frac{U}{T_0} F_1 \right) \Bigg], \end{aligned} \tag{2.39}$$
where
$$\mathrm{L} = \frac{1}{T_0} \frac{\mathrm{d}}{\mathrm{d}y} T_0 \frac{\mathrm{d}}{\mathrm{d}y} - \frac{2U'}{U} \frac{\mathrm{d}}{\mathrm{d}y} - \bar{\alpha}^2 \left(1 - \frac{U^2 M^2}{T_0} \cos^2\theta \right) \tag{2.40}$$
is the linear compressible Rayleigh operator.

Our interest is in the limiting form of the solutions to (2.36)–(2.40) near the critical level where $U(y) = 0$, which, without loss of generality, we can suppose to lie at $y = 0$. The expansion for Π_1 can be obtained in a standard way by the method of Frobenius. This was done in I for the case when the critical point coincides with one of the generalized inflexion points where $(U'/T_0)' = 0$. It was shown by Lees & Lin (1946) that this is a necessary condition for the existence of a subsonic neutral mode. The critical point is then a regular singular point for the compressible Rayleigh equation and the neutral mode is said to be regular. For supersonic modes the critical point need not coincide with a generalized inflexion point. Logarithmic terms must then be included in the Frobenius series to obtain two linearly independent solutions to (2.36). This is the singular neutral mode solution. The two linearly independent solutions to (2.36) in the singular case are

$$\tilde{\Pi}^{(1)} = 1-\tfrac{1}{2}\bar{\alpha}^2y^2+\tfrac{1}{3}\bar{\alpha}^2\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)y^3\ln|y|-\tfrac{1}{4}\bar{\alpha}^2\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)^2y^4\ln|y|+a_4y^4+\dots \tag{2.41}$$

and
$$\tilde{\Pi}^{(2)} = y^3-\frac{3}{4}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)y^4+\dots \tag{2.42}$$

as $y\rightarrow 0$, where

$$a_4 = \tfrac{1}{4}\bar{\alpha}^2\left(\frac{T''_c}{T_c}-\frac{2}{3}\frac{U'''_c}{U'_c}-\left(\frac{T'_c}{T_c}\right)^2+\frac{1}{2}\left(\frac{U''_c}{U'_c}\right)^2-\tfrac{1}{2}\bar{\alpha}^2-\frac{U'^2_cM^2\cos^2\theta}{T_c}+\frac{11}{12}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)^2\right). \tag{2.43}$$

Then
$$\Pi_1 = \frac{U'_c}{T_c}\left[\tilde{\Pi}^{(1)}+\tfrac{1}{3}\bar{\alpha}^2\left\{b_1^{\pm}+\frac{1}{2}\frac{U''_c}{U'_c}\right\}\tilde{\Pi}^{(2)}\right], \tag{2.44}$$

where $b_1^{\pm}$ are constants, $\pm$ refers to the regions above/below the critical layer and the jump $b_1^+-b_1^-$ will be determined by the flow in the critical layer. Note that for the regular modes $b_1^+ = b_1^-$.

Using (2.40)–(2.43) in (2.37) and the corresponding equations for Φ_1, Θ_1 and Ψ_1 shows that

$$\begin{aligned}F_1 = &-\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\ln|y|-\left(b_1^{\pm}+\frac{1}{3}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\right)-\left(\frac{U''_c}{U'_c}\right)\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)y\ln|y|\\ &-\left[\frac{T''_c}{T_c}-\frac{U'''_c}{U'_c}-\left(\frac{T'_c}{T_c}\right)^2+\left(\frac{U''_c}{U'_c}\right)^2-\bar{\alpha}^2-\frac{U'^2_cM^2\cos^2\theta}{T_c}+\frac{U''_c}{U'_c}b_1^{\pm}+\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\left(\frac{T'_c}{T_c}-\frac{7}{6}\frac{U''_c}{U'_c}\right)\right]y+\dots,\end{aligned} \tag{2.45}$$

$$\Phi_1 = 1-\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)y\ln|y|+\left[\frac{2}{3}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)-b_1^{\pm}\right]y+\dots, \tag{2.46}$$

$$\begin{aligned}\Theta_1 = \frac{T'_c}{U'_c\cos\theta}\Bigg[&\frac{1}{y}-\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\ln|y|\\ &+\left\{\frac{2}{3}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)-b_1^{\pm}+\frac{T''_c}{T'_c}-\frac{U''_c}{2U'_c}\right\}\Bigg]+(k-1)M^2U'_c\cos\theta+\dots\end{aligned} \tag{2.47}$$

and
$$\Psi_1 = \tan\theta\left[-\frac{1}{y}+\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\ln|y|+\frac{1}{3}\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)+b_1^{\pm}+\frac{U''_c}{2U'_c}-\frac{T'_c}{T_c}\right]+\dots \tag{2.48}$$

as $y\rightarrow 0^{\pm}$.

The general solution to (2.37) can be written as

$$A^{\dagger}\Pi_3 = \frac{2\mathrm{i}U_c'}{T_c\,\bar{\alpha}\cos\theta}\left(U_c\cos\theta\frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1}-\mathrm{i}S_1A^{\dagger}\right)\tilde{\Pi}_{P,1}-2\mathrm{i}\bar{\alpha}\frac{U_c'}{T_c}\frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1}\tilde{\Pi}_{P,2}, \tag{2.49}$$

where
$$\tilde{\Pi}_{P,1} = \Pi_{P,1}+c_{2,1}\tilde{\Pi}^{(1)}+\tfrac{1}{3}\bar{\alpha}^2\left(b_{2,1}^{\pm}+\frac{1}{2}\frac{U_c''}{U_c'}\right)\tilde{\Pi}^{(2)} \tag{2.50}$$

and
$$\tilde{\Pi}_{P,2} = \Pi_{P,2}+c_{2,2}\tilde{\Pi}^{(1)}+\tfrac{1}{3}\bar{\alpha}^2\left(b_{2,2}^{\pm}+\frac{1}{2}\frac{U_c''}{U_c'}\right)\tilde{\ }^{(2)}, \tag{2.51}$$

$b_{2,n}^{\pm}$ and $c_{2,n}$ are constants, and $\Pi_{P,1}$ and $\Pi_{P,2}$ are particular solutions of

$$\mathrm{L}\Pi_{P,1} = \frac{T_c}{U_c'}\left[\frac{U'}{U^2}\frac{\mathrm{d}}{\mathrm{d}y}+\frac{U}{T_0}(M\bar{\alpha}\cos\theta)^2\right]\Pi_1, \tag{2.52}$$

and
$$\mathrm{L}\Pi_{P,2} = T_c\left[1-\frac{(MU\cos\theta)^2}{T_0}\right]\frac{\Pi_1}{U_c'}, \tag{2.53}$$

whose behaviour near $y = 0^{\pm}$ can be found from (2.41) and (2.42) and the method of variation of parameters, but which are in general unbounded at infinity.

Using these in (2.39) gives that

$$A^{\dagger}F_3 = \frac{e_0(\bar{x}_1)}{y}+e_1(\bar{x}_1)\ln|y|+e_2(\bar{x}_1)+2\mathrm{i}\bar{\alpha}\frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1}\left(b_{2,2}^{\pm}-\frac{1}{\bar{\alpha}^2}b_1^{\pm}\right)$$
$$-\frac{2\mathrm{i}}{\bar{\alpha}\cos\theta}\left(U_c\cos\theta\frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1}-\mathrm{i}S_1A^{\dagger}\right)\left(b_{2,1}^{\pm}-\frac{U_c''}{2U_c'^2}b_1^{\pm}\right)+\dots \tag{2.54}$$

as $y\to 0^{\pm}$.

The constants appearing in the solutions (2.44), (2.50) and (2.51) can, in principle, be determined by applying appropriate boundary conditions at infinity. These require that the subsonic modes decay exponentially while the supersonic modes are outgoing. An alternative condition can be derived which does not require a complete solution to the $O(\epsilon^{\frac{7}{5}})$ problem for the pressure but only its local behaviour near the critical layer. The procedure has been outlined by Redekopp (1977), among others, and when applied to the present problem yields

$$b_{2,1}^{+}-b_{2,1}^{-}-(b_1^{+}-b_1^{-})c_{2,1}-\frac{U_c''}{2U_c'^2}(b_1^{+}-b_1^{-}) = \lim_{R\to\infty}\left[-\frac{T_c}{\bar{\alpha}^2}⨍_{-R}^{R}\left[\frac{T_0\Pi_1'^2}{U^3}+\frac{\bar{\alpha}^2T_0\Pi_1^2}{U^3}\right.\right.$$
$$+\frac{1}{2U_c'}\left\{\left(\frac{T_0'}{T_0}-\frac{2U'}{U}\right)\frac{\bar{\alpha}^2T_0\Pi_1^2}{U^2}+\frac{2\bar{\alpha}^2T_0}{U^2}\left(1-\frac{U^2M^2\cos^2\theta}{T_0}\right)\Pi_1\Pi_1'\right\}$$
$$\left.+\frac{\bar{\alpha}^2}{U_c'^2}\left(\frac{T_c'}{2T_c}-\frac{U_c''}{U_c'}\right)\frac{\mathrm{d}}{\mathrm{d}y}\frac{\Pi_1^2T_0}{U}\right]\mathrm{d}y+\frac{U_c'}{\bar{\alpha}^2}\left[\frac{T_0}{U^2}\left(\Pi_1\tilde{\Pi}_{P,1}'-\tilde{\Pi}_{P,1}\Pi_1'+\frac{T_c}{U_c'}\frac{\Pi_1\Pi_1'}{U}\right.\right.$$
$$\left.\left.\left.+\frac{T_c}{2U_c'^2}\bar{\alpha}^2\left(1-\frac{U^2M^2\cos^2\theta}{T_0}\right)\Pi_1^2+\frac{T_c\bar{\alpha}^2}{U_c'^3}\left(\frac{T_c'}{2T_c}-\frac{U_c''}{U_c'}\right)\Pi_1^2U\right)\right]\Bigg|_{-R}^{R}\right] \tag{2.55}$$

and

$$b_{2,2}^{+}-b_{2,2}^{-}-\frac{1}{\bar{\alpha}^2}(b_1^{+}-b_1^{-})-(b_1^{+}-b_1^{-})c_{2,2}$$
$$= \lim_{R\to\infty}\left[\frac{T_c}{\bar{\alpha}^4}⨍_{-R}^{R}\frac{T_0\Pi_1'^2}{U^2}\mathrm{d}y+\frac{U_c'}{\bar{\alpha}^2}\left[\frac{T_0}{U^2}\left(\Pi_1\tilde{\Pi}_{P,2}'-\Pi_1'\tilde{\Pi}_{P,2}-\frac{T_c}{U_c'\bar{\alpha}^2}\Pi_1\Pi_1'\right)\right]\Bigg|_{-R}^{R}\right], \tag{2.56}$$

where $⨍$ denotes the Cauchy principle value integral.

It is worth noting some of the differences between the outer solutions for the singular and regular problems. Equations (2.45) and (2.48) show that there is now a jump in the 'streamwise' and 'crossflow' velocity components at $O(\epsilon)$ in addition to the one at $O(\epsilon^{\frac{7}{5}})$. Also the $O(\epsilon^{\frac{7}{5}})$ 'streamwise' velocity component now has an algebraic singularity at the critical level from equation (2.54).

3. The critical layer

As shown in I the appropriate scaled transverse coordinate in the critical layer is

$$Y = y/\epsilon^{\frac{2}{5}}. \tag{3.1}$$

The form of the expansions of the critical-layer solution is obtained by re-expanding the inner limit of the outer solution obtained in the previous section in terms of this variable. These expansions, which provide the matching conditions for the critical-layer solution, are given in Appendix A. They suggest that the critical-layer solution has an expansion of the form

$$\begin{aligned}\bar{u} = \epsilon^{\frac{2}{5}}U_c' \cos\theta Y + \epsilon^{\frac{4}{5}}[\tfrac{1}{2}U_c'' \cos\theta Y^2 + a_c \lambda \bar{x}_1 \cos^2\theta] + \epsilon\tilde{u}_1 \\ + \epsilon^{\frac{6}{5}}[\tilde{u}_2 + a_c\lambda\bar{z}\sin\theta\cos\theta] + \epsilon^{\frac{7}{5}}\tilde{u}_3 + \dots,\end{aligned} \tag{3.2}$$

$$\bar{v} = -\epsilon^{\frac{3}{5}}\bar{\alpha}\,\mathrm{Re}\,[\mathrm{i}A^{\dagger}\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{4}{5}}\lambda\hat{V}_c + \epsilon\tilde{v}_1 + \dots, \tag{3.3}$$

$$\bar{w} = -U_c\sin\theta - \epsilon^{\frac{2}{5}}U_c'\sin\theta Y + \epsilon^{\frac{3}{5}}\tilde{w}_0 + \epsilon^{\frac{4}{5}}\tilde{w}_1 + \epsilon\tilde{w}_2 + \epsilon^{\frac{6}{5}}[\tilde{w}_3 - a_c\lambda\bar{z}\sin^2\theta] + \dots, \tag{3.4}$$

$$T = T_c + \epsilon^{\frac{2}{5}}T_c' Y + \epsilon^{\frac{3}{5}}\tilde{T}_0 + \epsilon^{\frac{4}{5}}\tilde{T}_1 + \epsilon\tilde{T}_2 + \dots, \tag{3.5}$$

$$p = 1 + \epsilon\tilde{p}_1(\zeta, \bar{x}_1) + \epsilon^{\frac{6}{5}}\tilde{p}_2(\zeta, \bar{x}_1) + \epsilon^{\frac{7}{5}}\tilde{p}_3 + \dots, \tag{3.6}$$

where we have put $\bar{v} = \epsilon^{-\frac{2}{5}}v$.

The momentum and energy equations inside the critical layer, to the required order of approximation, can be written as

$$\frac{\bar{\mathrm{D}}\bar{u}}{\mathrm{D}t} = -\frac{T}{kM^2p}\left(\frac{\partial p}{\partial\zeta} + \epsilon^{\frac{2}{5}}\frac{\partial p}{\partial\bar{x}_1}\right) + \frac{\lambda\epsilon^{\frac{2}{5}}T}{p}\left[T^n\frac{\partial^2\bar{u}}{\partial Y^2} + nT^{n-1}\frac{\partial T}{\partial Y}\frac{\partial\bar{u}}{\partial Y}\right], \tag{3.7}$$

$$\frac{\bar{\mathrm{D}}\bar{v}}{\mathrm{D}t} = -\frac{\epsilon^{-\frac{4}{5}}T}{kM^2p}\frac{\partial p}{\partial Y}, \tag{3.8}$$

$$\frac{\bar{\mathrm{D}}\bar{w}}{\mathrm{D}t} = \frac{\lambda\epsilon^{\frac{2}{5}}T}{p}\left[T^n\frac{\partial^2\bar{w}}{\partial Y^2} + nT^{n-1}\frac{\partial T}{\partial Y}\frac{\partial\bar{w}}{\partial Y}\right] \tag{3.9}$$

and
$$\frac{\bar{\mathrm{D}}T}{\mathrm{D}t} - \frac{(k-1)T}{kp}\frac{\bar{\mathrm{D}}p}{\mathrm{D}t} = M^2(k-1)\frac{\lambda T^{n+1}}{p}\epsilon^{\frac{2}{5}}(\bar{u}_Y^2 + \bar{w}_Y^2) + \frac{\lambda\epsilon^{\frac{2}{5}}T}{p\sigma_0}\frac{\partial}{\partial Y}(T^nT_Y), \tag{3.10}$$

where we have put

$$\frac{\bar{\mathrm{D}}}{\mathrm{D}t} = \left(\bar{u} - \epsilon^{\frac{2}{5}}\frac{S_1}{\bar{\alpha}}\right)\frac{\partial}{\partial\zeta} + \epsilon^{\frac{2}{5}}(\bar{u} + U_c\cos\theta)\frac{\partial}{\partial\bar{x}_1} + \bar{v}\frac{\partial}{\partial Y} + \bar{w}\frac{\partial}{\partial\bar{z}} \tag{3.11}$$

and subscripts denote partial derivatives.

The ultimate aim of the critical-layer analysis is to calculate the velocity jump across the critical layer which, when matched with the velocity jump from the outer solution, will yield an equation for the nonlinear evolution of the instability wave amplitude. To this end it is convenient to use the 'spanwise' vorticity

$$\Omega = -\frac{1}{\epsilon^{\frac{2}{5}}}\bar{u}_Y + \epsilon^{\frac{2}{5}}\left(\frac{\partial}{\partial \zeta} + \epsilon^{\frac{2}{5}}\frac{\partial}{\partial \bar{x}_1}\right)\bar{v} \tag{3.12}$$

as a dependent variable. The critical-layer vorticity equation can be written as

$$\frac{\bar{\mathrm{D}}\Omega}{\mathrm{D}t} - \frac{\Omega}{kp}\frac{\bar{\mathrm{D}}p}{\mathrm{D}t} - \Omega\bar{w}_{\bar{z}} = -\frac{\epsilon^{-\frac{2}{5}}}{kM^2 p}[(T_\zeta + \epsilon^{\frac{2}{5}}T_{\bar{x}_1})p_Y - T_Y(p_\zeta + \epsilon^{\frac{2}{5}}p_{\bar{x}_1})] - \frac{M^2(k-1)}{p}$$
$$\times \lambda\epsilon^{\frac{2}{5}}\Omega T^n(\bar{u}_Y^2 + \bar{w}_Y^2) - \frac{\lambda\epsilon^{\frac{2}{5}}\Omega}{p\sigma_0}\frac{\partial}{\partial Y}(T^n T_Y) - \lambda\frac{T}{p}\frac{\partial^2}{\partial Y^2}(T^n\bar{u}_Y) - \frac{\lambda T_Y}{p}\frac{\partial}{\partial Y}(T^n\bar{u}_Y) + \frac{1}{\epsilon^{\frac{2}{5}}}\bar{w}_Y\bar{u}_{\bar{z}}. \tag{3.13}$$

Substituting the expansions (3.2)–(3.6) into (3.7)–(3.13) we obtain a sequence of partial differential equations which can be written as

$$\left[U_c\cos\theta\frac{\partial}{\partial \bar{x}_1} + \left(U_c'\cos\theta Y - \frac{S_1}{\bar{\alpha}}\right)\frac{\partial}{\partial\zeta} - \frac{\lambda T_c^{n+1}}{\sigma_0}\frac{\partial^2}{\partial Y^2}\right]\bar{T}_{n-1} = \mathscr{T}_{n-1} \tag{3.14}$$

and
$$\left[U_c\cos\theta\frac{\partial}{\partial \bar{x}_1} + \left(U_c'\cos\theta Y - \frac{S_1}{\bar{\alpha}}\right)\frac{\partial}{\partial\zeta} - \lambda T_c^{n+1}\frac{\partial^2}{\partial Y^2}\right]\tilde{Q}_n = \mathscr{Q}_n, \tag{3.15}$$

for $n = 1, 2, 3$ where we have put

$$\bar{T}_0 = \tilde{T}_0, \quad \bar{T}_1 = \tilde{T}_1 - \tfrac{1}{2}T_c''Y^2 + t_1^{(1)}\bar{x}_1, \tag{3.16a, b}$$

$$\bar{T}_2 = \tilde{T}_2 - \frac{k-1}{k}T_c\tilde{p}_1 - \frac{T_c'}{U_c'\cos\theta}\tilde{u}_1 + t_2^{(1)}\,\mathrm{Re}\,[A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}], \tag{3.16c}$$

$$\tilde{Q}_1 = \tilde{u}_{1Y}, \quad \tilde{Q}_2 = \tilde{u}_{2Y} - \frac{U_c'\cos\theta}{T_c}\tilde{T}_1 + q_2^{(1)}\bar{x}_1 + q_2^{(2)}Y^2, \tag{3.17a, b}$$

$$\tilde{Q}_3 = \tilde{u}_{3Y} - \bar{\alpha}^2\,\mathrm{Re}\,[A^\dagger\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] - \frac{U_c'\cos\theta}{k}\tilde{p}_1 - \frac{U_c''}{U_c'}\tilde{u}_1 + q_3^{(1)}\,\mathrm{Re}\,[A^\dagger\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}], \tag{3.17c}$$

and the inhomogeneous terms $\mathscr{T}_{n-1}$ and $\mathscr{Q}_n$ along with the constants $t_1^{(1)}$, $q_2^{(1)}$ etc. are given in Appendix B. Since $\sigma_0 \neq 1$ the linear differential operators for the vorticity and temperature equations are not identical. We also note that when $1/\sigma_0 \neq n$ we need to obtain the solution for the temperature to $O(\epsilon)$ in order to determine the $O(\epsilon^{\frac{7}{5}})$ velocity jump across the critical layer (see equation (B 6)).

Integrating (3.17*a, c*) and using the matching conditions in Appendix A shows that

$$\int_{-\infty}^{+\infty}\tilde{Q}_1\,\mathrm{d}Y = -(b_1^+ - b_1^-)\,\mathrm{Re}\,[A^\dagger\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] \tag{3.18}$$

and

$$\int_{-\infty}^{+\infty}\tilde{Q}_3\,\mathrm{d}Y = \mathrm{Re}\left\{\left[\frac{-2\mathrm{i}}{\bar{\alpha}\cos\theta}\left(U_c\cos\theta\frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1A^\dagger\right)(b_{2,1}^+ - b_{2,1}^-)\right.\right.$$
$$\left.\left. + 2\mathrm{i}\bar{\alpha}\frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1}\left(b_{2,2}^+ - b_{2,2}^- - \frac{1}{\bar{\alpha}^2}(b_1^+ - b_1^-)\right)\right]\mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right\}. \tag{3.19}$$

Equation (3.19) will yield an equation for $A^\dagger$ once the solutions to (3.14) and (3.15) are obtained.

To facilitate the solution we introduce the following normalized variables:

$$\bar{x} = -\tfrac{1}{2}S_1 U_c' \bar{x}_1 - \bar{x}_0, \tag{3.20}$$

$$\eta = -2\bar{\alpha}(Y - S_1/(\bar{\alpha}U_c' \cos\theta))/S_1 U_c, \tag{3.21}$$

$$X = \bar{\alpha}\zeta - X_0, \tag{3.22}$$

$$A = 4\bar{\alpha}^2 A^{\dagger} e^{iX_0}/(U_c S_1)^2 U_c', \tag{3.23}$$

$$Q^{(n)} = \bar{\alpha}^2 \cos\theta \tilde{Q}_n / S_1^2 U_c U_c', \tag{3.24}$$

where the coordinate shifts $\bar{x}_0$ and X_0 are chosen so that

$$A \to \exp(\bar{\kappa}\bar{x} + i\phi_0) \quad \text{as} \quad \bar{x} \to -\infty \tag{3.25}$$

and ϕ_0 is a real phase shift which will be specified later.

For the purpose of deriving the equation for the instability wave amplitude from (3.19) we need only determine the fundamental component of $Q^{(3)}$. Furthermore, we need only consider those terms in the solution which make non-zero contributions to the integral in (3.19). To accomplish this we look for solutions of the form

$$Q_n = \mathrm{Re}\left[\sum_{m=-\infty}^{\infty} Q_m^{(n)}(\bar{x}, \eta)\, e^{imX}\right] \tag{3.26}$$

and

$$\bar{T}_n = \mathrm{Re}\left[\sum_{m=-\infty}^{\infty} T_m^{(n)}(\bar{x}, \eta)\, e^{imX}\right]. \tag{3.27}$$

The differential equations for the harmonic components, obtained by substituting (3.26) and (3.27), along with (3.20)–(3.24) into (3.14) and (3.15), are solved by successive integration of the differential equations in the same way as in I using the Fourier transform method of Hickernell (1984).

Introducing the Fourier transform pair

$$\hat{F}(K) = \int_{-\infty}^{\infty} e^{-iK\eta} F(\eta)\, d\eta, \quad F(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iK\eta} \hat{F}(K)\, dK, \tag{3.28}$$

the first few of these solutions are found to be

$$\hat{T}_1^{(0)} = \frac{-i\pi T_c' U_c S_1}{\bar{\alpha}\cos\theta} \exp(\bar{\lambda}K^3/3\sigma_0) H(-K) A(\bar{x}+K), \tag{3.29}$$

$$\hat{Q}_1^{(1)} = \left\{\frac{i\pi\bar{\alpha}\cos\theta}{S_1}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\exp(\bar{\lambda}K^3/3) - \frac{i\pi T_c' \bar{\alpha}\cos\theta}{S_1 T_c}\frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)}\right.$$
$$\left. \times \exp(\bar{\lambda}K^3/3)[1 - \exp(\bar{\lambda}K^3(1/\sigma_0 - 1)/3)]\right\} H(-K) A(\bar{x}+K), \tag{3.30}$$

$$\hat{T}_0^{(1)} = \frac{-i\pi T_c' U_c S_1}{2\bar{\alpha}\cos^2\theta}\int_{-\infty}^{\bar{x}} \exp(-\bar{\lambda}K^2[3(\bar{x}-\tilde{x}) - K]/3\sigma_0) A^*(\tilde{x}) A(\tilde{x}+K) K H(-K)\, d\tilde{x} \tag{3.31}$$

and

$$\tilde{Q}_0^{(2)} = \frac{i\pi\bar{\alpha}}{2S_1}\int_{-\infty}^{\bar{x}} \exp(-\bar{\lambda}K^2[3(\bar{x}-\tilde{x}) - K]/3) A^*(\tilde{x}) A(\tilde{x}+K) K H(-K)$$
$$\times \left\{\frac{T_c'}{T_c} - \frac{U_c''}{U_c'} - \frac{T_c'}{T_c}\frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)} + \frac{T_c'}{T_c}\left[1 + \frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)}\right]\right.$$
$$\left. \times \exp(-\bar{\lambda}(1/\sigma_0 - 1)K^2[3(\bar{x}-\tilde{x}) - K]/3) + \frac{T_c'}{T_c}\exp(\bar{\lambda}(1/\sigma_0 - 1)K^3/3)\right\} d\tilde{x}, \tag{3.32}$$

where $H(K)$ is the Heaviside step function and

$$\bar{\lambda} = \frac{-8\lambda T_c^{n+1}\bar{\alpha}^2}{U_c' \cos\theta (U_c S_1)^3} \tag{3.33}$$

is a normalized viscous parameter.

From (3.30) we find that

$$\int_{-\infty}^{\infty} Q_1^{(1)} \,\mathrm{d}\eta = \frac{\mathrm{i}\pi\bar{\alpha}\cos\theta}{2S_1}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) A(\bar{x}). \tag{3.34}$$

Then comparing (3.34) and (3.18) and using (3.20)–(3.24) gives the $O(\epsilon)$ jump in the streamwise velocity

$$b_1^+ - b_1^- = \mathrm{i}\pi\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right). \tag{3.35}$$

The complete solution for $\hat{Q}_1^{(3)}$ is fairly complicated. However, most of the terms in the solution make no contribution to the integral in (3.19). The only terms which make non-zero contributions to the velocity jump are the linear terms and terms resulting from the interaction of the mean flow change with the fundamental as in I. Here there are additional linear terms proportional to $((T_c'/T_c) - (U_c''/U_c'))$ and the additional nonlinear term from the mean flow change – fundamental interaction in the solution for the $O(\epsilon)$ temperature. When only these terms are substituted into (3.19) using (3.20)–(3.24) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} Q_1^{(3)} \,\mathrm{d}\eta = {} & \frac{\pi}{2U_c'}\left(\frac{T_c'''}{T_c} - \frac{U_c'''}{U_c'}\right) - \left(\frac{2U_c''}{U_c'} - 2U_c' c_{2,1}\right)\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\left(\tfrac{1}{2}U_c U_c' \cos\theta \frac{\mathrm{d}A}{\mathrm{d}\bar{x}} + \mathrm{i}A\right) \\ & - \tfrac{1}{2}\pi U_c' \bar{\alpha}^2 \cos\theta c_{2,2}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\frac{\mathrm{d}A}{\mathrm{d}\bar{x}} - \frac{\mathrm{i}\pi\bar{\alpha}(3(T_c'/T_c) - (U_c''/U_c'))}{4S_1\cos\theta} \\ & \times \int_{-\infty}^{\bar{x}}\int_{-\infty}^{\tilde{x}} \exp\left(-\bar{\lambda}(\bar{x}-\tilde{x})^2[3(\bar{x}-\hat{x}) - (\bar{x}-\tilde{x})]/3\right) \\ & \times A(\tilde{x})A(\hat{x})A^*(\hat{x}+\tilde{x}-\bar{x})(\bar{x}-\tilde{x})^2 \mathscr{K}(\bar{x},\tilde{x},\hat{x})\,\mathrm{d}\hat{x}\,\mathrm{d}x,\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\left(3\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\mathscr{K} = {} & \frac{T_c'}{T_c}\left[1 - \frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)}\right] - \frac{U_c''}{U_c'} + \frac{T_c'}{T_c} \\ & \times \exp\left(-\bar{\lambda}(1/\sigma_0 - 1)(\bar{x}-\tilde{x})^2[3(\bar{x}-\hat{x}) - 2(\bar{x}-\tilde{x})]/3\right) \\ & + \frac{T_c'}{T_c}\exp\left(\bar{\lambda}(1/\sigma_0 - 1)(\bar{x}-\tilde{x})^3/3\right) + \frac{T_c'}{T_c}\frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)} \\ & \times \exp\left(-\bar{\lambda}(1/\sigma_0 - 1)(\bar{x}-\tilde{x})^2[3(\bar{x}-\hat{x}) - (\bar{x}-\tilde{x})]/3\right).\end{aligned} \tag{3.37}$$

Comparing these with (3.19) and using (3.20)–(3.24) yields the amplitude evolution equation

$$\begin{aligned}\frac{\gamma}{\bar{\kappa}}A_{\bar{x}} = \gamma A - \int_{-\infty}^{\bar{x}}\int_{-\infty}^{\tilde{x}} & \exp\left(-\bar{\lambda}(\bar{x}-\tilde{x})^2[3(\bar{x}-\hat{x}) - (\bar{x}-\tilde{x})]/3\right) \\ & \times A(\tilde{x})A(\hat{x})A^*(\hat{x}+\tilde{x}-\bar{x})(\bar{x}-\tilde{x})^2\mathscr{K}(\bar{x},\tilde{x},\hat{x})\,\mathrm{d}\hat{x}\,\mathrm{d}\tilde{x},\end{aligned} \tag{3.38}$$

where

$$\gamma=\frac{2S_1\cos\theta}{\bar{\alpha}U'_c[(3T'_c/T_c)-(U''_c/U'_c)]}\left[\frac{T''_c}{T_c}-\frac{U'''_c}{U'_c}-\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\left(2\frac{U''_c}{U'_c}+\mathrm{i}\pi c_{2,1}\right)+\frac{2\mathrm{i}U'_c}{\pi}(b^+_{2,1}-b^-_{2,1})\right] \tag{3.39}$$

and

$$\frac{1}{\kappa}=\left\{\frac{2S_1\cos\theta\bar{\alpha}}{\pi\gamma[(3T'_c/T_c)-(U''_c/U'_c)]}\left[b^+_{2,2}-b^-_{2,2}-\mathrm{i}\pi\left(\frac{T'_c}{T_c}-\frac{U''_c}{U'_c}\right)\left(c_{2,2}+\frac{1}{\bar{\alpha}^2}\right)\right]+\tfrac{1}{2}\mathrm{i}U_c\right\}U'_c\cos\theta \tag{3.40}$$

are complex constants which are fully determined by (2.55), (2.56) and the solution to the linear, inviscid Rayleigh problem.

Equation (3.38) is the desired result.

3.1. *Limiting forms of the amplitude equation*

There are a number of limiting forms of equation (3.38) which are of particular interest and we discuss these cases individually.

3.1.1. *Case 1:* $\sigma_0 = n = 1$

In analyses of compressible flows the Prandtl number is often taken to be unity for simplicity. Taking $n = 1$ in (2.5) corresponds to the Chapman viscosity law. For this case we have

$$\mathscr{K} = 1. \tag{3.41}$$

The amplitude evolution equation then becomes identical to that obtained in I but the complex coefficients have the generalized definitions (3.39) and (3.40). Therefore the amplitude equation derived in I also describes the evolution of supersonic modes for $\sigma_0 = n = 1$ if (3.45) and (3.46) of I are replaced by (3.39) and (3.40). The subsonic case is recovered if we let $T'_c/T_c = U''_c/U'_c$.

3.1.2. *Case 2:* $\sigma_0 = 1, n$ *arbitrary*

For this case we obtain

$$\mathscr{K} = 1-\bar{n}\bar{\lambda}(\bar{x}-\tilde{x})^2[3(\bar{x}-\hat{x})-(\bar{x}-\tilde{x})], \tag{3.42}$$

where

$$\bar{n}=\frac{T'_c}{3T_c}\frac{\bar{\lambda}(1-n)}{[3(T'_c/T_c)-(U''_c/U'_c)]}. \tag{3.43}$$

The power-law dependence of viscosity on temperature produces an algebraic viscous term in addition to the exponential one. Note that this term also appears for the case of constant viscosity ($n = 0$) and only vanishes for the Chapman viscosity law ($n = 1$). For subsonic modes we have that

$$\bar{n}=\tfrac{1}{6}(1-n) \tag{3.44}$$

and, since n is less than one for realistic viscosity laws, $\bar{n}$ is always positive. For supersonic modes $\bar{n}$ can be either positive or negative.

4. Asymptotic solutions to the amplitude equation

The results of I and preliminary numerical solutions of (3.38) show that the ultimate state of the solution for the amplitude equation can be of two types.

If the viscous parameter $\bar{\lambda}$ is zero or sufficiently small the solution develops a singularity at a finite downstream distance. The source of this large increase in the instability wave amplitude is the vorticity production due to compressibility effects. The appearance and form of the singularity indicate that the present scaling eventually breaks down for these cases and that the subsequent evolution is governed by the full Euler equations (see I). For certain parameter ranges this singularity can be eliminated if $\bar{\lambda}$ is large enough and the solution goes to a finite-amplitude equilibrium.

The forms of these asymptotic solutions are now described.

4.1. *Singular solution*

For the singular case the asymptotic solution of (3.38) is (to leading order) the same as that given in I, which is

$$A = \frac{a}{(\bar{x}_s - \bar{x})^{\frac{5}{2}+\mathrm{i}\sigma}}, \quad \bar{x} \to \bar{x}_s, \tag{4.1}$$

where $\bar{x}_s$ and σ are real constants and a is a complex constant.

The constants σ and a are determined from the real and imaginary parts of

$$\frac{D(\sigma)}{\frac{5}{2}+\mathrm{i}\sigma} = \frac{-\gamma}{|a|^2\bar{\kappa}}, \tag{4.2}$$

where

$$D(\sigma) = \int_1^\infty \frac{(v-1)^2}{v^{\frac{5}{2}+\mathrm{i}\sigma}} \int_v^\infty \frac{\mathrm{d}u\,\mathrm{d}v}{u^{\frac{5}{2}+\mathrm{i}\sigma}(u+v-1)^{\frac{5}{2}-\mathrm{i}\sigma}}. \tag{4.3}$$

4.2. *Equilibrium solution*

The exponential term in the amplitude equation serves to damp out the history effects of the convolution integral and make the solution a more local one. As $\bar{x} \to \infty$, if the solution does not first become singular, the integral is dominated by the contributions near $\bar{x}$ and the evolution equation becomes

$$\frac{1}{\bar{\kappa}}\frac{\mathrm{d}A}{\mathrm{d}\bar{x}} = A - \frac{\frac{1}{2}(\frac{2}{3})^{\frac{2}{3}}\Gamma(\frac{1}{3})}{\gamma\bar{\lambda}^{\frac{4}{3}}} A|A|^2 Q, \tag{4.4}$$

where

$$\left(3\frac{T_c'}{T_c} - \frac{U_c'''}{U_c'}\right)Q = \frac{T_c'}{T_c}\left[1 - \frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)} + \left(\frac{2}{3-1/\sigma_0}\right)^{\frac{1}{3}} + \sigma_0\left(\frac{2}{1/\sigma_0+1}\right)^{\frac{1}{3}} + \sigma_0^{\frac{4}{3}}\frac{(1/\sigma_0 - n)}{(1/\sigma_0 - 1)}\right] - \frac{U_c'''}{U_c'}. \tag{4.5}$$

We therefore have that

$$|A|^2 \to \frac{\mathrm{Re}\,[\bar{\kappa}\bar{\lambda}^{\frac{4}{3}}]}{\frac{1}{2}(\frac{2}{3})^{\frac{2}{3}}\Gamma(\frac{1}{3})\,Q\,\mathrm{Re}\,(\bar{\kappa}/\gamma)}, \tag{4.6}$$

and

$$\mathrm{Im}\left(\frac{A'}{A}\right) \to \left|\frac{\bar{\kappa}}{\gamma}\right|^2 \frac{\mathrm{Im}\,(\gamma)}{\mathrm{Re}\,(\bar{\kappa}/\gamma)}, \tag{4.7}$$

provided

$$\mathrm{Re}\,(\bar{\kappa}/\gamma)\,Q > 0. \tag{4.8}$$

The limiting forms of Q corresponding to the special cases in §3.1 are

$$Q = 1 \quad \text{for} \quad \sigma_0 = n = 1,$$

and $$Q = 1-4\bar{n} \quad \text{for} \quad \sigma_0 = 1, \quad \text{arbitrary } n.$$

Equation (4.8) is a necessary condition for the existence of an equilibrium solution to (3.38). In order for the amplitude to reach the equilibrium solution when (4.8) is satisfied the normalized viscous parameter $\bar{\lambda}$ must be large enough to damp out the history effects in the integral of (3.38). The ultimate form of the solution for any given case must be found by numerical solution of the amplitude equation.

5. Numerical results and discussion

In I the normalized variables $A/(|\gamma|^{\frac{1}{2}}|\bar{\kappa}|^2)$ and $|\bar{\kappa}|\,\bar{x}-\bar{x}_0$ were introduced with $\bar{x}_0$ and the phase factor ϕ_0 in (3.25) is chosen so that

$$\frac{\bar{\kappa}}{|\bar{\kappa}|}\bar{x}_0+\mathrm{i}\phi_0 = \log|\gamma|^{\frac{1}{2}}|\bar{\kappa}|^2 \tag{5.1}$$

to show that the solutions to (3.38) can be characterized by the real parameters arg $\bar{\kappa}$, arg γ and $\bar{\lambda}/|\bar{\kappa}|^3$ for subsonic modes when $\sigma_0 = 1$ and $n = 1$. Numerical solutions were presented in I for $\ln[|A|/(|\gamma|^{\frac{1}{2}}|\bar{\kappa}|^2)]$ and the real and imaginary parts of $A'|\bar{\kappa}|/A$ *vs.* $|\bar{\kappa}|\,\bar{x}-\bar{x}_0$ for a number of combinations of arg $\bar{\kappa}$, arg γ and $\bar{\lambda}/|\bar{\kappa}|^3$. Note that the solutions presented in I are in terms of these normalized variables rather than those shown on the captions.

If instead the normalized varibles,

$$B(\tau) = \frac{\exp(-\bar{\kappa}(\bar{x}-\bar{x}_0)/2\bar{\kappa}_r)\,|\bar{\kappa}|^{\frac{1}{2}}A(\bar{x})}{|\gamma|^{\frac{1}{2}}(2\bar{\kappa}_r)^{\frac{5}{2}}}, \tag{5.2}$$

$$\tau = 2\bar{\kappa}_r\,\bar{x}-\bar{x}_0 \tag{5.3}$$

are introduced, where $\bar{\kappa}_r = \mathrm{Re}(\bar{\kappa})$, with $\bar{x}_0$ and ϕ_0 chosen so that

$$\frac{\bar{\kappa}\bar{x}_0}{2\bar{\kappa}_r}+\mathrm{i}\phi_0 = \ln\left(\frac{|\gamma|^2(2\bar{\kappa}_r)^{\frac{5}{2}}}{|\bar{\kappa}|^2}\right), \tag{5.4}$$

it is found that the solutions to (3.38) can be characterized by two real parameters $\arg(\bar{\kappa}/\gamma)$ and $\hat{\lambda} = \bar{\lambda}/(2\bar{\kappa}_r)^3$ when $\sigma_0 = 1$ and $n = 1$ for both subsonic and supersonic modes. In particular this shows that the inviscid solutions to the amplitude equation can be completely characterized by a one-parameter family of curves for both subsonic and supersonic modes. These solutions are shown as the solid curves in figure 1 for a number of values of $\arg(\bar{\kappa}/\gamma)$. The dashed curves are the asymptotic solutions from (4.1). For convenience in presenting the viscous solutions below we have plotted $\ln|B|+\frac{1}{2}\tau$. We show solutions only for $\pi < \arg(\bar{\kappa}/\gamma) < 0$ since $B(\tau,[\bar{\kappa}/\gamma]^*) = B^*(\tau,\bar{\kappa}/\gamma)$.

All the inviscid solutions end in a singularity at a finite downstream distance except the case $\arg(\bar{\kappa}/\gamma) = 0$ which we will show to be a physically unrealizable parameter value. For small values of $\arg(\bar{\kappa}/\gamma)$ the solution goes to the asymptotic one through a series of oscillations whose amplitude decreases as the singularity is approached. These oscillations are symptomatic of an energy exchange between the fundamental instability wave and the mean flow, higher harmonics generated by a critical-layer nonlinearity, or both. At larger values of $\arg(\bar{\kappa}/\gamma)$ the nonlinear term always augments the growth of the fundamental.

In figure 2 a set of solutions is shown for the case when $\arg(\bar{\kappa}/\gamma) = \frac{1}{6}\pi$ with $\bar{n} = 0$

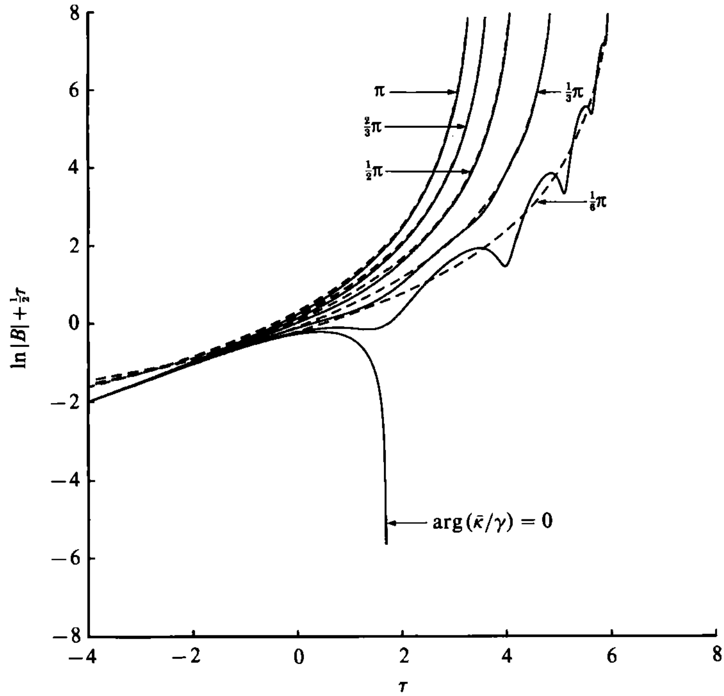

FIGURE 1. Normalized instability wave amplitude $B(\tau)$ *vs.* τ for various $\arg(\bar{\kappa}/\gamma)$, $\hat{\lambda} = 0$.

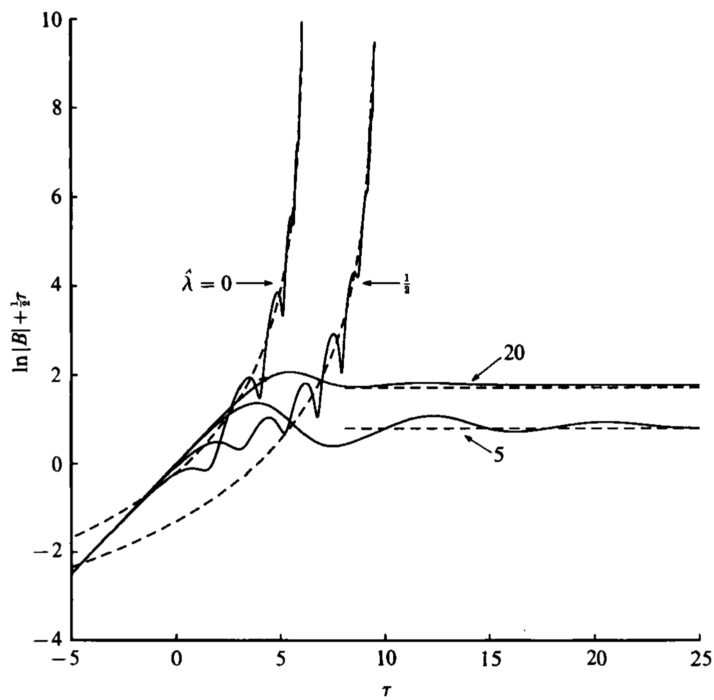

FIGURE 2. Normalized instability wave amplitude $B(\tau)$ *vs.* τ for various $\hat{\lambda}$, $\arg(\bar{\kappa}/\gamma) = \frac{1}{6}\pi$, $\bar{n} = 0$.

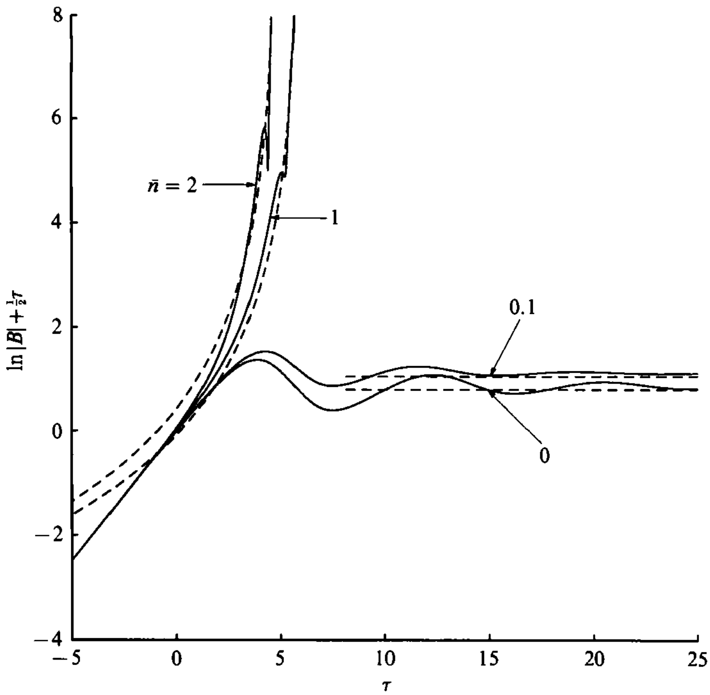

FIGURE 3. Normalized instability wave amplitude $B(\tau)$ *vs.* τ for various $\bar{n}$, $\arg(\bar{\kappa}/\gamma) = \frac{1}{6}\pi$, $\hat{\lambda} = 5$.

for a number of values of $\hat{\lambda}$. For this case an equilibrium solution is possible and the numerical results show that it is obtained when the normalized viscous parameter is sufficiently large.

Figures 3–5 show the dependence of the solution on the normalized viscosity–temperature law parameter $\bar{n}$. They show that the ultimate form of the solution can be quite different depending on the sign and magnitude of $\bar{n}$. Figure 3 shows a case where the solution reaches the equilibrium solution when $\bar{n} = 0$ but becomes singular as $\bar{n}$ is increased. Figures 4 and 5 show that a solution which is singular for $\bar{n} = 0$ can reach the equilibrium solution with larger $\bar{n}$ but only if $\hat{\lambda}$ is sufficiently large.

The parameters appearing in the amplitude evolution equation can be calculated from (2.55), (2.56), (3.39), (3.40) and the solution to the linear, inviscid Rayleigh problem (2.36) with appropriate boundary conditions. It is of particular interest to know, given a set of physical parameters, whether the necessary condition for the existence of an equilibrium solution, given by (4.8), is met.

For the linear calculations a transformation was made to the Howarth–Dorodnitzyn variable

$$\tilde{\eta} = \int_0^y \frac{1}{T_0}\mathrm{d}y. \tag{5.5}$$

For $\sigma_0 = 1$ and $n = 1$ the mean momentum and energy equations decouple and the temperature can be expressed in terms of the velocity $\bar{U}(\tilde{\eta})$ by the Crocco relation

$$T_0 = [\tfrac{1}{2}(k-1)M^2\bar{U}(\tilde{\eta})+\beta_T]\{1-\bar{U}(\tilde{\eta})\}+\bar{U}(\tilde{\eta}), \tag{5.6}$$

where β_T is the ratio of the slow-stream to the fast-stream temperatures. In all the calculations we have set $k = 1.4$ and considered only parameter values for which the

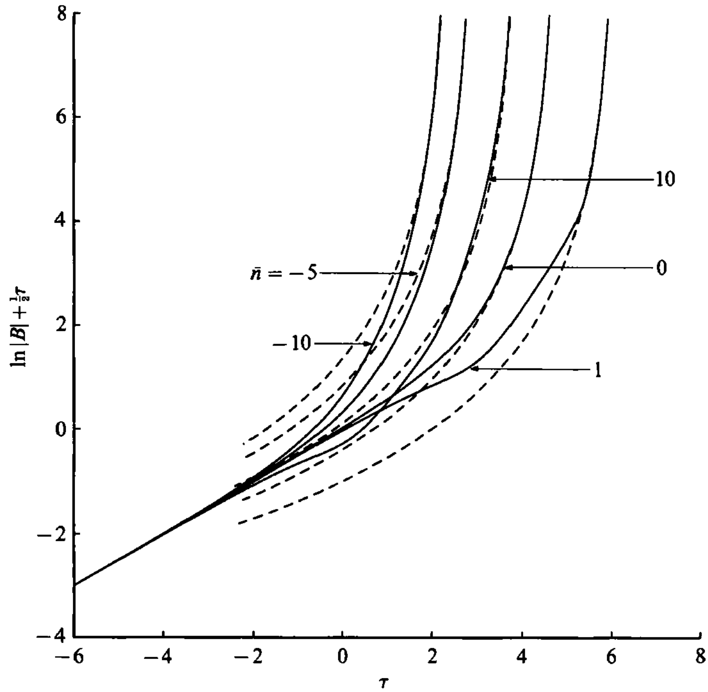

FIGURE 4. Normalized instability wave amplitude $B(\tau)$ *vs.* τ for various $\bar{n}$, $\arg(\bar{\kappa}/\gamma)=\frac{2}{3}\pi$, $\hat{\lambda}=5$.

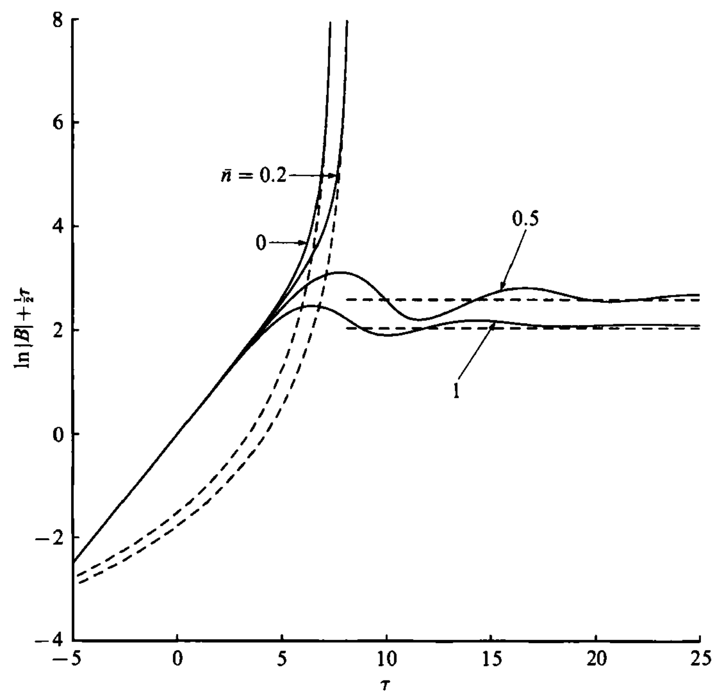

FIGURE 5. Normalized instability wave amplitude $B(\tau)$ *vs.* τ for various $\bar{n}$, $\arg(\bar{\kappa}/\gamma)=\frac{2}{3}\pi$, $\hat{\lambda}=50$.

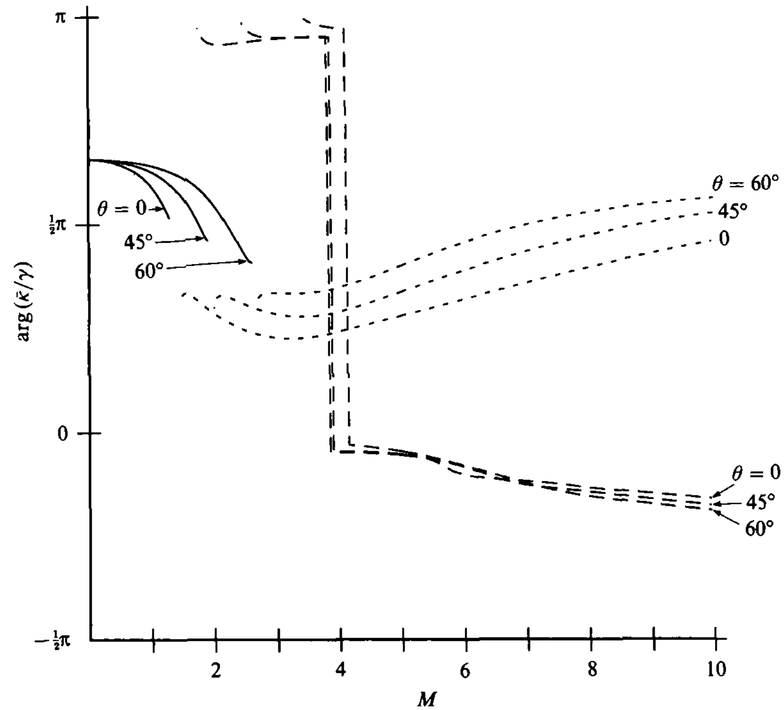

FIGURE 6. Arg$(\bar{\kappa}/\gamma)$ *vs.* Mach number for $\beta_T = \frac{1}{2}$ and various obliqueness angles, tanh profile: subsonic (solid), slow supersonic (short dashed) and fast supersonic (long dashed) modes.

shear layer is convectively unstable (Pavithran & Redekopp 1989; Jackson & Grosch 1990).

In the first set of results presented we take the mean velocity to be a tanh in the $\tilde{\eta}$ variable:

$$\bar{U}(\tilde{\eta}) = \tfrac{1}{2}(1+\tanh(\tilde{\eta})). \tag{5.7}$$

In figures 6–8 values of the real parameter arg $(\bar{\kappa}/\gamma)$ are plotted as a function of Mach number for a number of obliqueness angles with $\beta_T = \frac{1}{2}$, 2 and 1 respectively. The solid curves are results for the subsonic modes while those for the slow and fast supersonic modes are given by the short and long dashed curves, respectively. Recall that the necessary condition for an equilibrium solution to exist when $\sigma_0 = n = 1$ is $-\frac{1}{2}\pi < \arg(\bar{\kappa}/\gamma) < \frac{1}{2}\pi$.

For $\beta_T = \frac{1}{2}$, figure 6 shows that for the subsonic mode arg $(\bar{\kappa}/\gamma)$ is greater than $\frac{1}{2}\pi$ for Mach numbers less than about 2, decreases with increasing M and crosses over into the range where equilibration is possible at a Mach number which increases with obliqueness angle. For the slow supersonic mode, which is the most rapidly growing of the two supersonic modes at this temperature ratio, arg $(\bar{\kappa}/\gamma)$ lies between 0 and $\frac{1}{2}\pi$ over most of the Mach number range but exceeds $\frac{1}{2}\pi$ at the very high Mach numbers. Smaller obliqueness angles require a larger Mach number for this to occur. The behaviour of the fast supersonic mode is quite different, however. When this mode first appears arg $(\bar{\kappa}/\gamma)$ is between $\frac{1}{2}\pi$ and π and remains relatively constant as the Mach number is increased until there is a jump of π at a particular Mach number. At this Mach number $|\bar{\kappa}/\gamma|$ is zero and the nonlinear critical-layer effects vanish entirely. At this point there is a transition from the singular to a possible equilibrium

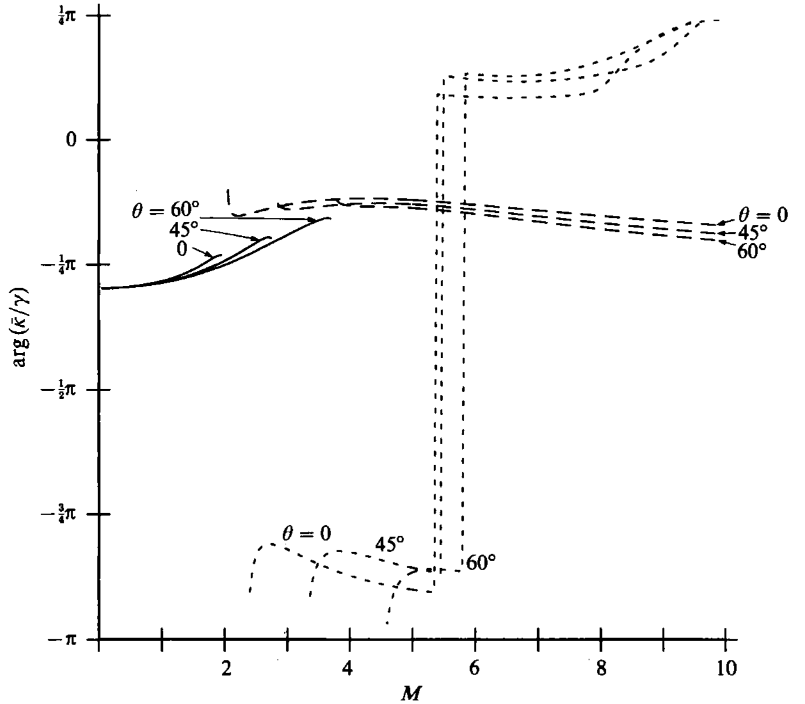

FIGURE 7. Arg $(\bar{\kappa}/\gamma)$ *vs.* Mach number for $\beta_T = 2$ and various obliqueness angles, tanh profile: subsonic (solid), slow supersonic (short dashed) and fast supersonic (long dashed) modes.

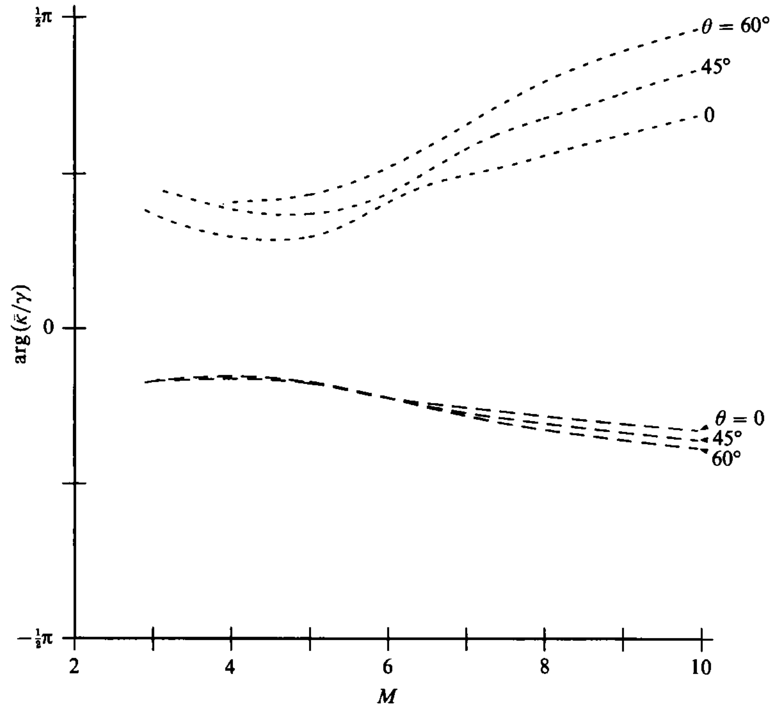

FIGURE 8. Arg $(\bar{\kappa}/\gamma)$ *vs.* Mach number for $\beta_T = 1$ and various obliqueness angles, tanh profile: slow supersonic (short dashed) and fast supersonic (long dashed) modes.

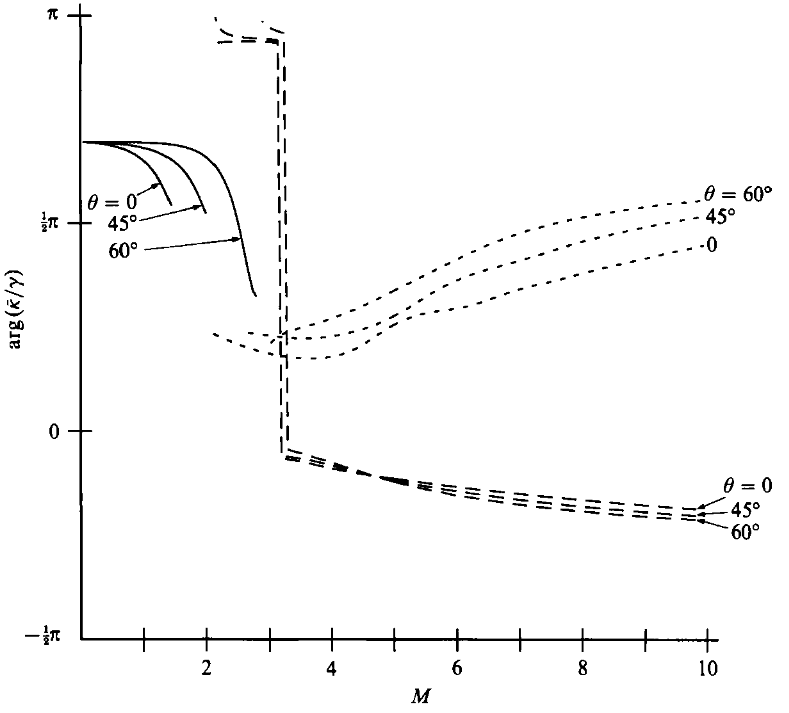

FIGURE 9. Arg ($\bar{\kappa}/\gamma$) *vs.* Mach number for $\beta_T = \frac{1}{2}$ and various obliqueness angles, Lock profile: subsonic (solid), slow supersonic (short dashed) and fast supersonic (long dashed) modes.

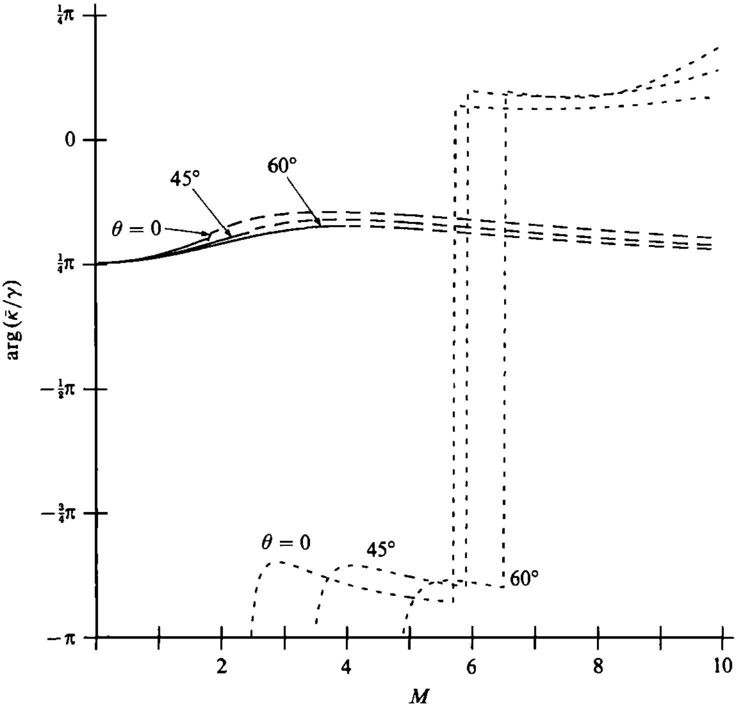

FIGURE 10. Arg ($\bar{\kappa}/\gamma$) *vs.* Mach number for $\beta_T = 2$ and various obliqueness angles, Lock profile: subsonic (solid), slow supersonic (short dashed) and fast supersonic (long dashed) modes.

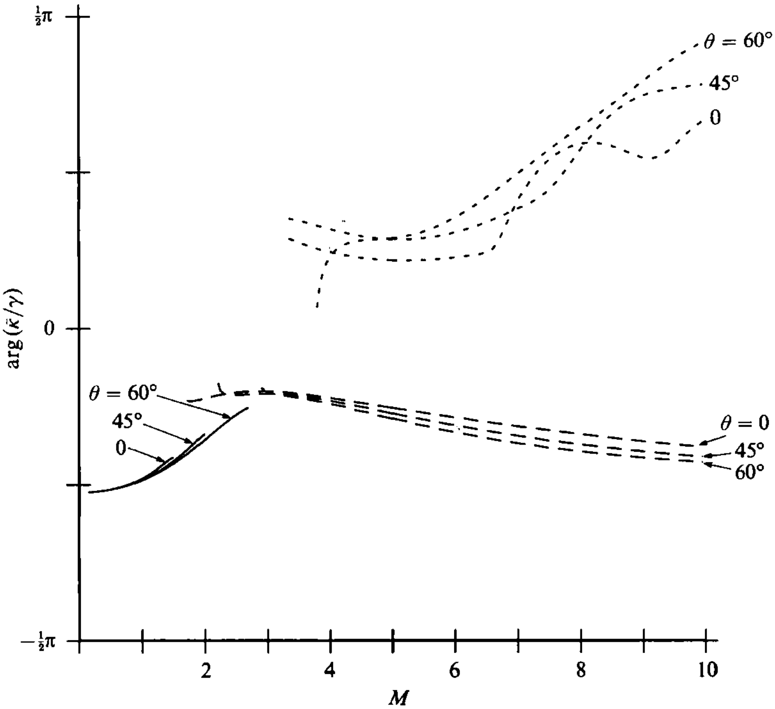

FIGURE 11. Arg$(\bar{\kappa}/\gamma)$ *vs.* Mach number for $\beta_T = 1$ and various obliqueness angles, Lock profile: subsonic (solid), slow supersonic (short dashed) and fast supersonic (long dashed) modes.

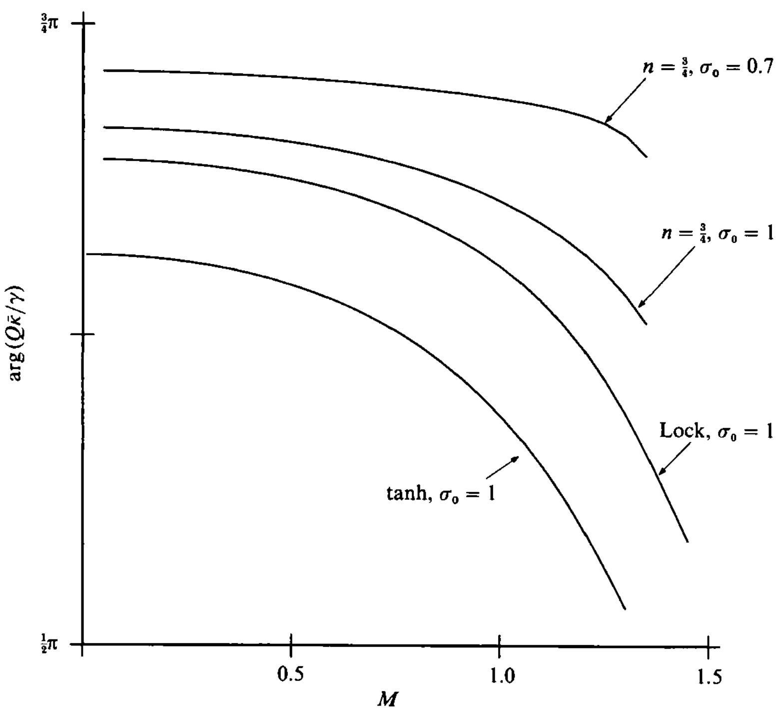

FIGURE 12. Generalized equilibrium criterion for two-dimensional subsonic mode, $\beta_T = \frac{1}{2}$ for different mean flow models.

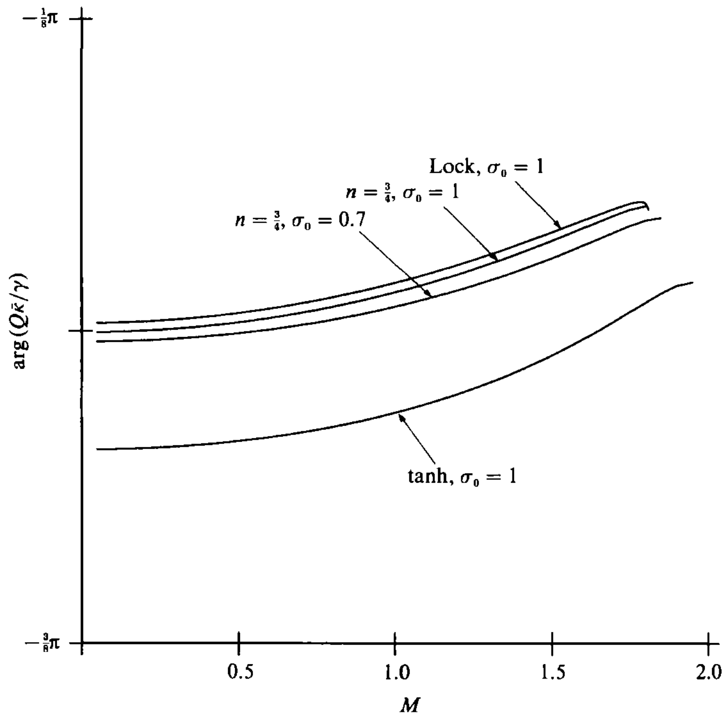

FIGURE 13. Generalized equilibrium criterion for two-dimensional mode, $\beta_T = 2$ for different mean flow models.

solution for the instability wave amplitude. The Mach number at which this occurs increases with increasing θ.

For $\beta_T = 2$ (figure 7) arg $(\bar{\kappa}/\gamma)$ is bounded between $-\frac{1}{2}\pi$ and 0 for the subsonic and fast supersonic modes (the latter being the most rapidly growing supersonic mode at this β_T). Since the absolute value of arg $(\bar{\kappa}/\gamma)$ is always less than $\frac{1}{6}\pi$ for these modes we would expect, from figure 1, to see oscillations in the amplitude to precede the inviscid singularity. For the slow supersonic mode no equilibrium solution exists until the Mach number exceeds a critical value (which increase with obliqueness angle) at which $|\bar{\kappa}/\gamma|$ is zero. This behaviour is similar to that of the fast supersonic mode when $\beta_T = \frac{1}{2}$.

When $\beta_T = 1$ the symmetry properties of the tanh-profile cause $|\bar{\kappa}/\gamma|$ to be zero for the subsonic mode at all obliqueness angles and hence there are no nonlinear effects of the type considered here. For the supersonic modes, however. the symmetry is broken and the nonlinear effects are present. Figure 8 shows that arg $(\bar{\kappa}/\gamma)$ lies between $\pm\frac{1}{2}\pi$ for the two supersonic modes over the entire Mach-number range calculated with positive values for the slow mode and negative for the fast.

The results in figures 6–8 indicate that, for the subsonic modes, for Mach numbers less than around 2, there is a critical temperature ratio below which equilibrium solutions do not exist and the explosive growth of the instability wave will always occur. For the tanh profile the critical temperature ratio is 1. The results for the most rapidly growing supersonic mode, on the other hand, show that an equilibrium solution is possible in all cases except at very high Mach numbers and low temperature ratios. Note that the exceptional case of arg $(\bar{\kappa}/\gamma) = 0$ is never encountered in any of the results obtained here.

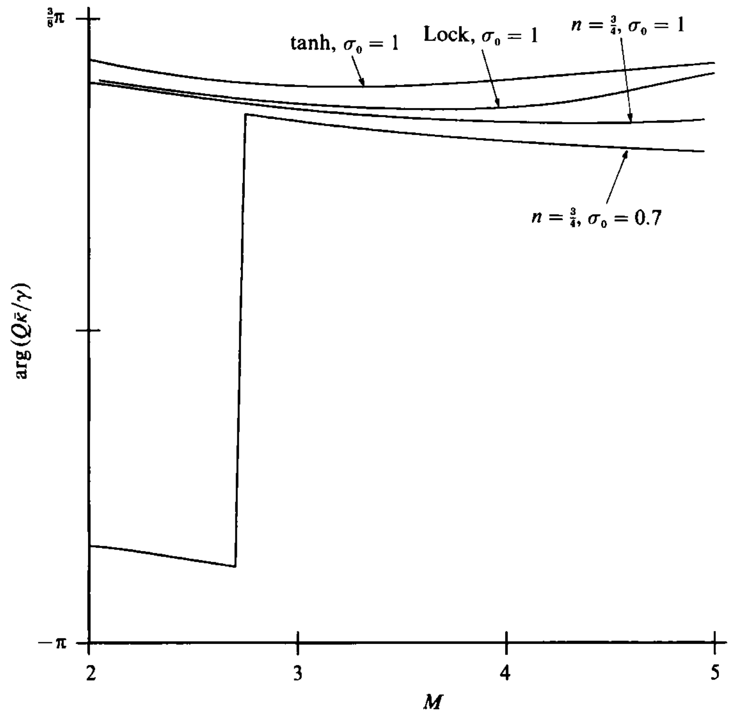

FIGURE 14. Generalized equilibrium criterion for two-dimensional slow supersonic mode, $\beta_T = \frac{1}{2}$ for different mean flow models.

In order to determine the effect of the mean flow model on the conclusions reached above we repeated some of the linear calculations using different models. In figures 9–11 we show results for the same cases as above but with the similarity solution of Lock (1951) for the mean velocity profile. As these figures show, the tanh and Lock profiles give results which are in qualitative agreement. The only significant difference is that the asymmetry of the Lock profile allows the subsonic modes to exhibit nonlinear effects for the $\beta_T = 1$ case. The value of $\arg(\bar{\kappa}/\gamma)$ lies between $-\frac{1}{2}\pi$ and 0 for these modes. This indicates that the critical temperature ratio for an equilibrium solution to exist for subsonic modes with Mach number less than about 2 is somewhat less than 1 for the Lock profile.

For Prandtl number different from unity and/or a viscosity law other than Chapman's the mean velocity and temperature profiles must be obtained by simultaneous solution of the mean momentum and energy equations. We have computed results for the case of Prandtl number of 0.7 and $n = \frac{3}{4}$. The latter is meant to approximate the Sutherland viscosity law (Schlichting 1979, p. 329).

In figures 12 and 13 the general equilibrium criterion is plotted for the two-dimensional subsonic mode with $\beta_T = \frac{1}{2}$ and 2 respectively, for four choices of mean flow model. These figures show that the different mean flow models produce qualitatively similar results for the subsonic mode. In particular, all the models predict that the equilibrium solution does not exist for $\beta_T = \frac{1}{2}$ but does exist for $\beta_T = 2$. The critical temperature ratio above which the equilibrium solution exists, however, depends on the mean velocity and temperature profiles used.

Figure 14 shows results using the same mean flow models for the slow supersonic two-dimensional mode with $\beta_T = \frac{1}{2}$. The results obtained by putting $\sigma_0 = 1$ are

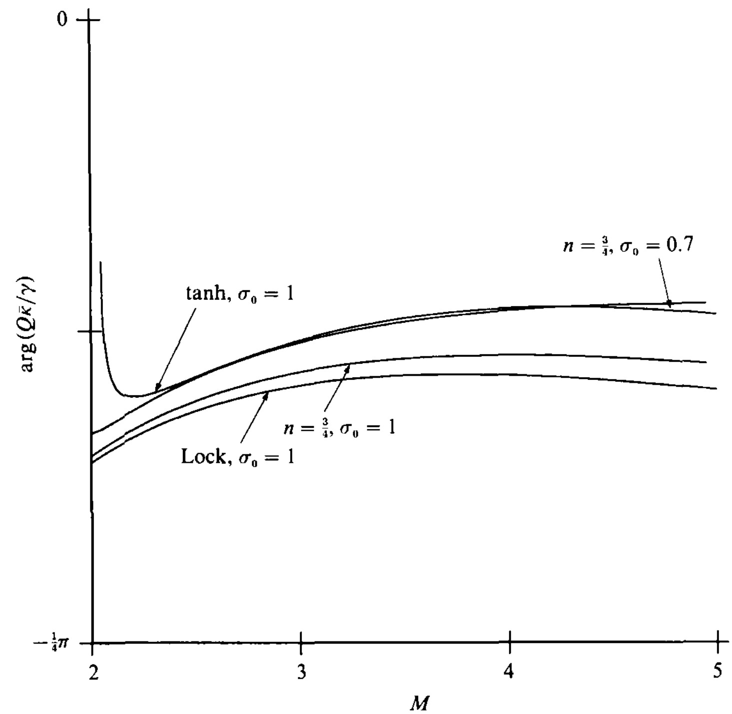

FIGURE 15. Generalized equilibrium criterion for two-dimensional fast supersonic mode, $\beta_T = 2$ for different mean flow models.

qualitatively similar over the Mach-number range shown but the case $\sigma_0 = 0.7$, $n = \frac{3}{4}$ behaves quite differently: the equilibrium solution cannot be obtained until the Mach number is about 2.75. The different behaviour for this mean flow model can perhaps be explained by the behaviour of the linear neutral solution for this case.

Jackson & Grosch (1989*b*) found that the temperature ratio at which there is a saddle point in the $(\bar{U}(\tilde{\eta}_c), M)$-plane plays an important role in the behaviour of the linear neutral solutions. They showed that the saddle-point temperature ratio depended on the mean flow model and, in particular, was strongly influenced by the Prandtl number. The value of $\beta_T = \frac{1}{2}$ is less than the saddle-point value for the tanh and Lock profiles with $\sigma_0 = 1$ (and also presumably for the $\sigma_0 = 1$, $n = \frac{3}{4}$ case) but is greater than the saddle-point value for $\sigma_0 = 0.7$ with the Sutherland viscosity law. Since the value of the saddle-point temperature ratio is related to the symmetry, or lack thereof, of the mean profiles at the critical point it also plays an important role in the nonlinear evolution of the instabilities.

Results for the fast supersonic two-dimensional mode with $\beta_T = 2$ are shown in figure 15. There is qualitative agreement for all the mean flow models in this case.

The results presented above indicate that, generally speaking, cooling the low-speed stream and decreasing the Mach number has a destabilizing influence on the nonlinear evolution of the subsonic modes since only the explosive-growth-type solution for the amplitude is possible. On the other hand increasing β_T and M stabilizes the evolution in the sense that an equilibrium solution becomes possible. The stabilizing/destabilizing influence of these parameters is similar to that found for the linear spatial theory (Gropengeisser 1969; Jackson & Grosch 1989*a*, *b*). For supersonic modes it is possible for the reverse to occur, as figures 6 and 7 show. Cooling the low-speed stream is stabilizing for the slow supersonic mode for Mach numbers up to about five.

The author would like to thank Drs M. E. Goldstein and Lennart S. Hultgren for helpful discussions on this work.

Appendix A. The inner limit of the outer solution

The inner limit of the outer solution in terms of the critical layer variable is given in this Appendix:

$$\begin{aligned}
\bar{u} = {} & \epsilon^{\frac{2}{5}} U_c' \cos\theta Y + \epsilon^{\frac{4}{5}}[\tfrac{1}{2}U_c'' \cos\theta Y^2 + a_c \lambda \bar{x}_1 \cos^2\theta] \\
& + \epsilon \operatorname{Re}\left[-\left(b_1^{\pm} + \frac{1}{3}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\right) - \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\ln|Y| + \frac{e_0(\bar{x})}{Y}\right] A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta} \\
& + \epsilon^{\frac{6}{5}}\left[\frac{U_c''' \cos\theta Y^3}{6} + a_c \lambda \bar{z} \sin\theta\cos\theta + a_c' Y \lambda \bar{x}_1 \cos^2\theta\right] \\
& + \epsilon^{\frac{7}{5}} \operatorname{Re}\left\{\left[-\frac{U_c''}{U_c'}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) Y \ln|Y| A^{\dagger}\right.\right. \\
& - \left(\frac{T_c''}{T_c} - \frac{U_c'''}{U_c'} - \bar{\alpha}^2 - \frac{U_c'^2 M^2 \cos^2\theta}{T_c} - \left(\frac{T_c'}{T_c}\right)^2 + \left(\frac{U_c''}{U_c'}\right)^2 \right. \\
& \left. + \frac{U_c''}{U_c'} b_1^{\pm} + \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\left(\frac{T_c'}{T_c} - \frac{7U_c''}{6U_c'}\right)\right) Y A^{\dagger} + e_1(\bar{x}_1)\ln|Y| + e_2(\bar{x}_1) \\
& - \frac{2\mathrm{i}}{\bar{\alpha}\cos\theta}\left(U_c \cos\theta \frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^{\dagger}\right)\left(b_{2,1}^{\pm} - \frac{U_c''}{2U_c'^2} b_1^{\pm}\right) \\
& \left.\left. + 2\mathrm{i}\bar{\alpha}\frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1}\left(b_{2,2}^{\pm} - \frac{1}{\bar{\alpha}^2} b_1^{\pm}\right)\right] \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right\} + \ldots,
\end{aligned} \tag{A 1}$$

$$v = -\epsilon\bar{\alpha} \operatorname{Re}[\mathrm{i}A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{8}{5}} \lambda \hat{V}_c + \ldots, \tag{A 2}$$

$$\begin{aligned}
\bar{w} = {} & -U_c \sin\theta - \epsilon^{\frac{2}{5}} U_c' \sin\theta Y - \epsilon^{\frac{3}{5}} \frac{\tan\theta}{Y} \operatorname{Re}[A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] - \epsilon^{\frac{4}{5}}[\tfrac{1}{2}U_c'' Y^2 \sin\theta \\
& + a_c \lambda \bar{x}_1 \cos\theta \sin\theta] - \epsilon^{\frac{6}{5}}[U_c''' Y^3 \tfrac{1}{6} \sin\theta + a_c \lambda \bar{z} \sin^2\theta + a_c' Y \lambda \bar{x}_1 \sin\theta\cos\theta] + \ldots,
\end{aligned} \tag{A 3}$$

$$\begin{aligned}
T = {} & T_c + \epsilon^{\frac{2}{5}} T_c' Y + \epsilon^{\frac{3}{5}} \frac{T_c'}{U_c' \cos\theta Y} \operatorname{Re}[A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \epsilon^{\frac{4}{5}} \tfrac{1}{2} T_c'' Y^2 \\
& + \epsilon \operatorname{Re}\left[\left\{\frac{T_c'}{U_c' \cos\theta}\left[-\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\ln|Y| + \frac{2}{3}\frac{T_c'}{T_c} - b_1^{\pm} + \frac{T_c''}{T_c'} - \frac{7}{6}\frac{U_c''}{U_c'}\right]\right.\right. \\
& \left.\left. + (k-1) M^2 \cos\theta U_c'\right\} A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right] + \ldots,
\end{aligned} \tag{A 4}$$

$$\begin{aligned}
p = {} & 1 + \epsilon k M^2 \cos\theta \frac{U_c'}{T_c} \operatorname{Re} A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta} + \epsilon^{\frac{7}{5}} k M^2 \operatorname{Re}\left\{\frac{2\mathrm{i}U_c'}{T_c \bar{\alpha}} c_{2,1}\left(U_c \cos\theta \frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^{\dagger}\right) \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right. \\
& \left. - 2\mathrm{i}\bar{\alpha} U_c' \frac{\cos\theta}{T_c} c_{2,2} \frac{\mathrm{d}A^{\dagger}}{\mathrm{d}\bar{x}_1} \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right\} \ldots.
\end{aligned} \tag{A 5}$$

Appendix B. The inhomogeneous terms and some constants in the critical-layer equations

The inhomogeneous terms in the critical-layer equations (3.14) and (3.15) are

$$\mathscr{T}_0 = \bar{\alpha} T_c' \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}], \tag{B 1}$$

$$\mathscr{Q}_1 = -U_c' \bar{\alpha} \cos\theta \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) \mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] - \lambda U_c' \cos\theta T_c^n \left(\frac{1}{\sigma_0} - n\right) \tilde{T}_{0YY}, \tag{B 2}$$

$$\mathscr{T}_1 = \bar{\alpha} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}]\, \tilde{T}_{0Y}, \tag{B 3}$$

$$\mathscr{Q}_2 = \bar{\alpha} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}]\, \tilde{Q}_{1Y} - 2U_c' \frac{\cos\theta \bar{\alpha}}{T_c} \tilde{T}_{0Y} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] + \lambda U_c' \cos\theta T_c^n \left(1 + n - \frac{2}{\sigma_0}\right) \tilde{T}_{1YY}, \tag{B 4}$$

$$\begin{aligned}\mathscr{T}_2 = {} & \bar{\alpha} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}]\, \tilde{T}_{1Y} - \hat{\mathrm{L}}\tilde{T}_0 + \frac{T_c' U_c''}{2U_c'} Y \bar{\alpha} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}] \\ & - 2M^2(k-1)\lambda T_c^{n+1} U_c' \sin\theta \tilde{w}_{0Y} + \frac{\lambda T_c^n T_c' Y(n+1)}{\sigma_0} \tilde{T}_{0YY} + \left(\frac{2}{\sigma_0} - 1\right) \lambda n T_c^n T_c' \tilde{T}_{0Y} \\ & + \frac{T_c T_c'}{U_c' \cos\theta k M^2} (\tilde{p}_{3\zeta} + \tilde{p}_{1\bar{x}_1}) - \frac{T_c' \lambda T_c^{n+1}}{U_c' \cos\theta} \left(1 - \frac{1}{\sigma_0}\right) \tilde{Q}_{1Y} \\ & + \frac{T_c'}{U_c' \cos\theta} \left(\frac{T_c'}{T_c} - \frac{3U_c''}{2U_c'} + \frac{T_c''}{T_c'}\right) \mathrm{Re}\left[\left(U_c \cos\theta \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^\dagger\right) \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right],\end{aligned} \tag{B 5}$$

$$\begin{aligned}\mathscr{Q}_3 = {} & \bar{\alpha} \,\mathrm{Re}\,[\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}]\, \tilde{Q}_{2Y} - \frac{T_c}{kM^2}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)(\tilde{p}_{3\zeta} + \tilde{p}_{1\bar{x}_1}) - \tfrac{1}{2} Y \bar{\alpha} U_c'' \cos\theta \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) \\ & \times \mathrm{Re}\,\mathrm{i}A^\dagger \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta} - \mathrm{L}\tilde{Q}_1 + \lambda \cos\theta \sin\theta (2M^2(k-1) U_c'^2 T_c^n - a_c)\, \tilde{w}_{0Y} \\ & + \left[\frac{T_c''}{T_c} - \frac{U_c'''}{U_c'} - \left(\frac{T_c'}{T_c}\right)^2 + \left(\frac{U_c''}{U_c'}\right)^2 + \left(\frac{T_c'}{T_c} - \frac{U_c''}{2U_c'}\right)\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\right] \\ & \times \mathrm{Re}\left[\left(U_c \cos\theta \frac{\mathrm{d}A^\dagger}{\mathrm{d}\bar{x}_1} - \mathrm{i}S_1 A^\dagger\right) \mathrm{e}^{\mathrm{i}\bar{\alpha}\zeta}\right] + \lambda U_c' \cos\theta T_c^n \left(\frac{1}{\sigma_0} - n\right)\left(\tilde{T}_{2YY} + \frac{T_c'}{U_c' \cos\theta} \tilde{Q}_{1Y}\right) \\ & + \lambda T_c^{n-1} \cos\theta Y \left(\frac{1}{\sigma_0} - n\right) [U_c'' T_c^n + n U_c' T_c'] \tilde{T}_{0YY} + \left[2\lambda U_c' \cos\theta n T_c^{n-1} T_c' \left(\frac{1}{\sigma_0} - n\right)\right. \\ & \left. - 4n\lambda T_c^n U_c'' \cos\theta\right] \tilde{T}_{0Y} - \lambda T_c^n T_c' [(n+1) Y \tilde{Q}_{1YY} + (2n+1) \tilde{Q}_{1Y}],\end{aligned} \tag{B 6}$$

where
$$\mathrm{L} = (\tfrac{1}{2} U_c'' \cos\theta Y^2 + a_c \bar{x}_1 \cos^2\theta) \frac{\partial}{\partial\zeta} + U_c' \cos\theta Y \frac{\partial}{\partial\bar{x}_1} + \hat{V}_c \lambda \frac{\partial}{\partial Y} \tag{B 7}$$

and
$$\hat{\mathrm{L}} = (\tfrac{1}{2} U_c'' \cos\theta Y^2 + a_c \bar{x}_1 \cos^2\theta) \frac{\partial}{\partial\zeta} + U_c' \cos\theta Y \frac{\partial}{\partial\bar{x}_1} + \frac{\hat{V}_c \lambda}{\sigma_0} \frac{\partial}{\partial Y} \tag{B 8}$$

are linear differential operators and $\tilde{w}_0$ is determined from the solution of

$$\mathrm{L}\tilde{w}_0 = -\bar{\alpha} U_c' \sin\theta \, \mathrm{Re}\,[\mathrm{i} A^{\dagger} \mathrm{e}^{\mathrm{i}\bar{\alpha}\xi}]. \tag{B 9}$$

The constants defined in (3.16) and (3.17) are

$$t_1^{(1)} = -\frac{\lambda}{U_c \cos\theta}\left[-\hat{V}_c T_c' + M^2(k-1)\, T_c^{n+1} U_c'^2 + \frac{n T_c^n T_c'^2}{\sigma_0} + U_c \sin^2\theta d_c + \frac{T_c^{n+1} T_c''}{\sigma_0}\right], \tag{B 10}$$

$$\begin{aligned} q_2^{(1)} = {} & \frac{\lambda U_c' T_c^{n+1}}{U_c}\left(\frac{T_c'''}{T_c} - \frac{U_c'''}{U_c'} - \left(\frac{T_c'}{T_c}\right)^2 + \left(\frac{U_c''}{U_c'}\right)^2 + \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\left(\frac{T_c'}{T_c} - \frac{U_c''}{2U_c'}\right)\right) \\ & + \frac{3\lambda U_c'' T_c^{n+1}}{2U_c}\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) - \frac{\lambda}{U_c}\left[U_c' \hat{V}_c\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right) - 2M^2(k-1)\, U_c'^3 T_c^n \right. \\ & - U_c' T_c'^2 n T_c^{n-1}\left(\frac{2}{\sigma_0} - n^2\right) + T_c^n T_c' U_c''(2n+1) - U_c \sin^2\theta d_c \\ & \left. + U_c' T_c''\left(1 - \frac{2}{\sigma_0} + n\right)\right], \end{aligned} \tag{B 11}$$

$$q_2^{(2)} = \tfrac{1}{2} U_c' \cos\theta\left(\frac{T_c'''}{T_c} - \frac{U_c'''}{U_c'} - \left(\frac{T_c'}{T_c}\right)^2 + \left(\frac{U_c''}{U_c'}\right)^2 + \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\left(\frac{T_c'}{T_c} - \frac{U_c''}{2U_c'}\right)\right) + \tfrac{3}{4} U_c'' \cos\theta\left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right), \tag{B 12}$$

$$t_2^{(1)} = -\frac{T_c'}{U_c' \cos\theta}\left(\frac{T_c'}{T_c} - \frac{3U_c''}{2U_c'} + \frac{T_c''}{T_c'}\right) \tag{B 13}$$

and

$$q_1^{(3)} = \frac{T_c'''}{T_c} - \frac{U_c'''}{U_c'} - \left(\frac{T_c'}{T_c}\right)^2 + \left(\frac{U_c''}{U_c'}\right)^2 + \left(\frac{T_c'}{T_c} - \frac{U_c''}{U_c'}\right)\left(\frac{T_c'}{T_c} - \frac{U_c''}{2U_c'}\right). \tag{B 14}$$

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